Instituto Superior de Ciências do Trabalho e da Empresa
Faculdade de Ciências da Universidade de Lisboa

Departamento de Finanças do ISCTE
Departamento de Matemática da FCUL

The valuation of turbo warrants under the CEV Model

Ana Domingues

Master's degree in

Mathematical Finance

2012

Instituto Superior de Ciências do Trabalho e da Empresa
Faculdade de Ciências da Universidade de Lisboa

Departamento de Finanças do ISCTE
Departamento de Matemática da FCUL

The valuation of turbo warrants under the CEV Model

Ana Domingues

Dissertation submitted for obtaining the Master's degree in

Mathematical Finance

Supervisors:

Professor João Pedro Vidal Nunes
Professor José Carlos Gonçalves Dias

2012

Abstract

This thesis uses the Laplace transform of the probability distributions of the minimum and maximum asset prices and of the expected value of the terminal payoff of a down-and-out option to derive closed-form solutions for the prices of lookback options and turbo call warrants, under the Constant Elasticity of Variance (CEV) and geometric Brownian motion (GBM) models. These solutions require numerical computations to invert the Laplace transforms. The analytical solutions proposed are implemented in *Matlab* and *Mathematica*. We show that the prices of these contracts are sensitive to variations of the elasticity parameter β in the CEV model.

Keywords: Turbo warrants, Lookback options, Constant Elasticity of Variance model, Laplace transform.

Resumo

O trabalho desenvolvido nesta tese é baseado no artigo de Wong and Chan (2008) e pretende determinar o preço de *turbo call warrants* segundo os modelos *Constant Elasticity of Variance* (CEV) e *geometric Brownian motion* (GBM). Este derivado financeiro é um contrato que tem o mesmo *payoff* que uma opção de compra *standard* se uma determinada barreira especificada não for tocada. Se o preço do activo subjacente tocar a barreira então um novo contrato começa com um novo *payoff* terminal (*rebate*) igual à diferença entre o valor mais baixo do activo registado durante o período especificado para este novo contrato e o preço de exercício.

Os *turbo warrants* são casos especiais de opções barreira uma vez que o *rebate* é calculado como outra opção exótica. Estes contratos apareceram primeiro no fim de 2001 na Alemanha e nessa altura eram apenas opções barreira *standard knock-out* com o nome ‘‘warrant’’. O mercado de *turbo warrants* tem tido grande evolução desde a sua introdução. Desde então, existem alguns autores que têm estudado este tipo de contratos segundo diversos modelos. Eriksson (2006) derivou fórmulas fechadas para o preço de *turbo warrants* quando o activo subjacente segue um processo lognormal. Eriksson and Persson (2006) compararam dois métodos distintos para calcular numericamente o preço dos contratos *turbo warrants* da sociedade Société Generale: o método de Monte Carlo e o método de diferenças finitas. A avaliação de *turbo warrants* segundo o modelo CEV foi implementada por Wong and Chan (2008). Estes autores também consideram um modelo cujo processo de volatilidade estocástica é *fast mean-reverting* e um modelo de volatilidade *time-scale*. Wong and Lau (2008) obtiveram soluções analíticas para os *turbo warrants* segundo o modelo de difusão *double exponential jump diffusion*.

O preço dos contratos *turbo warrants* pode ser calculado com base na avaliação de opções barreira e *lookback*, ambas opções *path-dependent*. O *payoff* final deste tipo de opções depende do preço do activo subjacente durante um certo período de tempo. As opções barreira podem terminar (*knock-out*) ou começar (*knock-in*) se o preço do activo subjacente atingir uma determinada barreira durante um certo período de tempo. Sendo assim existem oito opções barreira diferentes: opções de venda ou compra *up-and-out*,

down-and-out, *up-and-in* e *down-and-in*. Já o *payoff* das opções *lookback* depende do preço máximo ou mínimo do activo subjacente atingido durante a vida da opção. Existem dois tipos de opções *lookback*: opções *fixed strike lookback* e opções *floating strike lookback*. Enquanto que as primeiras possuem um preço de exercício pré-especificado, as segundas não têm.

O primeiro modelo estudado para a avaliação de opções barreira e opções *lookback* foi o modelo de Black and Scholes (1973). Este modelo revolucionou os mercados financeiros. No entanto, tem pressupostos que não são observados no mercado. O modelo de Black and Scholes (1973) assume que o preço do activo subjacente segue uma distribuição lognormal e a sua volatilidade é constante, mas o mercado mostra que isto não é verdade. A relação inversa entre o preço de exercício e a volatilidade implícita, conhecida por *volatility smile*, não é capturada por este modelo. Sendo assim, outros modelos foram estudados. O modelo CEV, que foi desenvolvido por Cox (1975), é um dos modelos que é consistente com a existência de correlação negativa entre os retornos e a volatilidade (*leverage effect*) e com a *volatility smile*. A comparação entre os modelos de Black and Scholes (1973) e de Cox (1975) foi feita por diversos autores. MacBeth and Merville (1980) concluiram que os preços obtidos com o modelo CEV eram mais próximos dos preços de mercado, especialmente quando o parâmetro de elasticidade do modelo é negativo. Boyle and Tian (1999) concluiram que a diferença de preços entre os dois modelos é maior para opções *path-dependent* do que para opções *standard*.

Esta tese mostra que para a avaliação de *turbo warrants* é necessário recorrer à avaliação de opções de compra *floating strike lookback* e de opções barreira de compra *down-and-out*. Seguimos os resultados de Davydov and Linetsky (2001) para calcular os preços das opções barreira e *lookback* e demonstramos os resultados relativos às opções *lookback*. Usamos a transformada de Laplace da distribuição de probabilidades dos preços mínimo e máximo do activo e do valor esperado do *payoff* final de uma opção barreira *single knock-out* para derivar fórmulas fechadas para o preço de opções *lookback* e de *turbo warrants*, segundo os modelos CEV e GBM. Utilizamos o algoritmo descrito por Abate and Whitt (1995) para inverter estas transformadas de Laplace. As soluções analíticas dos preços dos contratos foram implementadas nos programas *Matlab* e *Mathematica*.

A tese encontra-se estruturada da seguinte forma: no capitulo 1 encontra-se um resumo da literatura referente aos contratos referidos anteriormente: opções barreira, opções *lookback* e *turbo warrants* e um resumo da tese. No capitulo 2 são definidos os vários tipos de opções *lookback*, descritos os seus *payoffs* finais e é feita a avaliação deste tipo de opções antes da maturidade segundo um processo de difusão unidimensional geral. A mesma analise é feita para os contratos de *turbo warrants* no capitulo seguinte. No

capítulo 4, o modelo de Black-Scholes é introduzido e a as fórmulas fechadas para o preço das opções *lookback* e dos *turbo warrants* segundo este modelo são derivadas de duas formas diferentes: usando as formulas dos capítulos 2 e 3 e usando o formulário de Zhang (1998). Os preços destes derivados segundo o modelo CEV são determinados no capitulo 5, onde também é descrito este modelo. Os resultados numéricos encontram-se no capitulo 6, bem como um breve resumo do algoritmo proposto por Abate and Whitt (1995). As conclusões deste trabalho são apresentadas no capítulo 7.

Palavras-Chave: Turbo warrants, Opções lookback, Modelo de Constant Elasticity of Variance, transformada de Laplace.

Acknowledgements

First and foremost I would like to express my very great appreciation to my supervisors, Professor João Pedro Vidal Nunes and Professor José Carlos Gonçalves Dias, for theirs guidance and support throughout my thesis. I would also like to thank Professor João Pedro Vidal Nunes by the enthusiasm showed in their classrooms, that motivated me to finish the curricular component of the course and to write this thesis.

I would like to thank Ana Casimiro and Sónia Gomes for the friendship and help provided during the course.

Finally, I would like to thank Lia Domingues and Bruno Matos for their patience, support and encouragement throughout my course.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

A turbo call (put) warrant is a contract which payoff is the same that the standard call (put) option if a prespecified barrier has not been hit by the underlying asset price before maturity. If the underlying asset price hits the barrier a rebate is paid. For turbo call warrants the rebate is the difference between the lowest recorded stock-price during a prespecified period after the barrier is hit and the strike price, and for turbo put warrants the rebate is calculated as the difference between the strike price and the largest recorded stock-price during a prespecified period after the barrier is hit.

Then turbo warrants are special types of barrier options because the rebate is calculated as another exotic option. They first appeared in Germany at the end of 2001. Back then, these contracts were only standard knock-out barrier options with the name "warrant". The market segment for turbo warrants has had a big evolution since its introduction. On February 2005, Société Generale listed the first 40 turbo warrants on the Nordic Derivatives Exchange. In Asia, the Hong Kong Exchange and Clearing Limited introduced the callable bull/bear contracts, which are essentially turbo warrants.

Since then, there are some authors that studied these contracts under several models. Eriksson (2006) derived closed-form solutions for the price of turbo warrants when the underlying asset follows a lognormal process. Eriksson and Persson (2006) compared two different methods to price numerically the turbo warrants offered by the Société Generale: Monte Carlo and finite difference methods. The valuation of turbo warrants under the Constant Elasticity of Variance (CEV) model was implemented by Wong and Chan (2008). These authors also consider a fast mean-reverting stochastic volatility process and two time-scale volatility models. Further, Wong and Lau (2008) obtained analytical solutions for turbo warrants under the double exponential jump diffusion model in terms of Laplace transforms.

The price of turbo warrants can be obtained trough the valuation of barrier and lookback options. Barrier options are probably the oldest path-dependent options. The payoff of these options depends on the path of the underlying asset's price. A barrier option can become worthless or come into existence if the underlying asset price reaches a certain level during a certain period of time. Snyder (1969) describes down-and-out options as "limited risk special options". The lognormal assumption was covered by several authors. The first pricing solution for down-and-out calls under the Black and Scholes (1973) model[1] were derived by Merton (1973), and geometric Brownian motion assumption instituted the first step to the development of further studies on the pricing of barrier options. Rubinstein and Reiner (1991), Benson and Daniel (1991) and Hudson (1991) extended the analysis made by Merton (1973) for all eight types of single-barrier options[2]. Still under the usual, a binomial method for barrier options was studied by Boyle and Lau (1994). Kunitomo and Ikeda (1992) covered the valuation of double barrier options expressing the probability density as a sum of normal density functions. Expressions for the Laplace transform of a double barrier option price were derived by Geman and Yor (1996). Pelsser (2000) derived analytical solutions to price double barriers using a contour integration method. Further, Schroder (2000) derived the pricing formulas of Kunitomo and Ikeda (1992) inverting the Laplace transforms derived in Geman and Yor (1996). Based upon the Fourier series expansion, Lo and Hui (2007) proposed an approach for computing accurate estimates of Black-Scholes double barrier option prices with time-dependent parameters.

Although the Black and Scholes (1973) model revolutionized the financial markets, its underlying assumptions are not observed in the market. The Black and Scholes (1973) model assumes that the underlying asset price follows a lognormal distribution and its volatility is constant, but the market shows that this is not true. For example, Schmalensee and Trippi (1978) find a strong negative relationship between the stock prices changes and changes in implied volatility. Jackwerth and Rubinstein (2001) show that the usual geometric Brownian motion is unable to accommodate the negative skewness and the high kurtosis that are usually implicit in empirical asset return distributions. The inverse relation between the implied volatility and the strike price, known as *volatility smile* (see Dennis and Mayhew, 2002), is not captured by the Black and Scholes (1973) model, and therefore new models were studied.

The Constant Elasticity of Variance (CEV) model, which was developed by Cox (1975), is one model that is consistent with the existence of a negative correlation between stock returns and realized volatility, known as *leverage effect* (see Bekaert and Wu, 2000), and

[1]The Black and Scholes (1973) model assumes that the price of the underlying asset follows a geometric Brownian motion, and as a consequence the future underlying asset possesses a lognormal distribution.

[2]Up-and-out call or put, down-and-out call or put, up-and-in call or put and down-and-in call or put.

with the *volatility smile* (see Dennis and Mayhew, 2002). The pricing of barrier options under the CEV model is done by various authors. Boyle and Tian (1999) used a trinomial lattice to price single and double barrier options under this model. Davydov and Linetsky (2001) derived analytical formulae for the prices of double barriers and lookback options under the CEV model. Later, Davydov and Linetsky (2003) developed eigenfunctions expansions for single and double barrier options, which were used to invert the Laplace transforms of these contracts derived in Davydov and Linetsky (2001). Further, Lo et al. (2009) derived the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers.

Other models were used to price the barrier options. Under the Heston (1993) model, Lipton (2001) and Faulhaber (2002) propose two different methods to price continuously monitored double barrier options: the method of images and the eigenfunction expansion approach. However, they have to assume two unrealistic assumption: a zero *drift* for the underlying Itô process and the absence of correlation between the asset return and its volatility. Still considering the Heston (1993) model, Griebsch and Wystup (2008) priced discretely monitored barrier options through a multidimensional numerical integration approach, avoiding the previous two assumptions described. Kuan and Webber (2003) price single- and double-barrier options using the knowledge of the first passage time density of the underlying asset price to the barrier level(s) under the geometric Brownian motion assumption and one-factor interest rate models. However, the approach of Kuan and Webber (2003) is the least efficient of the approaches under the lognormal assumption (see Nunes and Dias, 2011). Under a one-dimensional diffusion process, Mijatović (2010) decomposed the price of double barrier options into a sum of integrals (along the time-dependent barriers) of the option's deltas. Finally, Nunes and Dias (2011) also decompose the double barrier option price into a sum of integrals but over the first passage time distributions of the (time-dependent) barriers. This last approach is based on a more general multifactor and Markovian financial model, provides efficient pricing solutions, and it is able to accommodate stochastic volatility, stochastic interest rates and endogenous bankruptcy.

As a barrier option, a lookback option is also a path-dependent option. Its payoff depends on the maximum or minimum stock price reached during the life of the option. Lookback options were first studied by Goldman, Sosin and Shepp (1979a) and Goldman, Sosin and Gatto (1979b) who derive closed-form pricing formulas under the lognormal assumption. These analytical pricing formulas are re-derived and extended by Conze and Viswanathan (1991). Babbs (2000) proposes a binomial model for floating strike lookback options under the Black-Scholes assumptions. Using the arbitrage arguments of the Cox et al. (1979) model, Cheuk and Vorst (1997) develope a binomial model for these options. Further, Bermin (2000) shows how Malliavin calculus can be used to derive the hedging strategy for any kind of path-dependent options, and in particular for lookback options. Like in barrier options, the pricing of lookback options under the CEV model is done by Boyle and Tian (1999) and Davydov and Linetsky (2001, 2003). Linetsky (2004) gives an analytical characterization of lookback option prices in terms of spectral expansions, in particular, under the CEV diffusion. Finally, Xu and Kwok (2005) derive general integral price formulas for lookback option models using partial differential equation techniques.

The work developed in this thesis is based on the paper of Wong and Chan (2008) and investigates the pricing of turbo warrants under the Black-Scholes and the CEV models. To compute the price of turbo warrants, we follow the results of Davydov and Linetsky (2001) for barrier and lookback options, including the proof of the results for the lookback contracts. Thus, we use the Abate and Whitt (1995) algorithm in *Matlab* and *Mathematica* to compute the necessary Laplace Transforms.

This thesis is organized as follow: In Chapter 2 we define the different types of lookback options and their terminal payoffs. The value of a lookback option before maturity is derived under a general one-dimensional diffusion process. The same analysis is done in Chapter 3 but for turbo warrants. In Chapter 4, the Black-Scholes model is introduced and its underlying assumptions are described. The formulae to compute the price of lookback options and turbo warrants is derived in two different ways: using the formulae of Zhang (1998) and using the formulas derived from the previous chapters 2 and 3. The price of lookback options and turbo warrants under the CEV model is explained in Chapter 5 and the CEV model is described. In Chapter 6, we present numeric results for lookback options and turbo warrants. The conclusions of this work are presented in Chapter 7.

Chapter 2

Lookback Options

European lookback options are exotic options whose terminal payoff depends on the maximum or on the minimum attained by the underlying asset price during the life of the option. There are two types of lookback options: floating and fixed strike lookback options.

The option's strike price of a floating strike lookback option is determined at maturity. A floating strike lookback call gives the option holder the right to buy at the lowest price recorded during the option's life and a floating strike lookback put gives the holder the right to sell at the highest price recorded during the option's life.

A fixed strike lookback option has a prespecified strike price. The payoff of a fixed strike call option is the maximum difference between the underlying asset's highest price (recorded during the option's life) and the strike or zero, whichever is greater. The payoff of a fixed strike put option is the maximum difference between the strike and underlying asset's lowest price (recorded during the option's life) or zero, whichever is greater.

In next definitions, S_T denotes the asset price at option's maturity T, while $m_{0,T}$ and $M_{0,T}$ denote the minimum and maximum asset prices recorded during the option's life, respectively.

Definition 1. *The time-T value of a floating strike lookback call option on the asset S, with a unit contract size, inception at time 0 and expiry date at time $T(>0)$ is:*

$$LC_{fl}(T; S_T, m_{0,T}, T) = S_T - m_{0,T} \,. \tag{2.1}$$

Definition 2. *The time-T value of a floating strike lookback put option on the asset S, with a unit contract size, inception at time 0 and expiry date at time $T(>0)$ is:*

$$LP_{fl}(T; S_T, M_{0,T}, T) = M_{0,T} - S_T \,. \tag{2.2}$$

Definition 3. *The time-T value of a fixed strike lookback call option on the asset S, with a unit contract size,a strike price equal to K, inception at time 0 and expiry date at time $T(>0)$ is:*

$$LC_{fx}(T; S_T, K, M_{0,T}, T) = (M_{0,T} - K)^+ \, . \tag{2.3}$$

Definition 4. *The time-T value of a fixed strike lookback put option on the asset S, with a unit contract size,a strike price equal to K, inception at time 0 and expiry date at time $T(>0)$ is:*

$$LP_{fx}(T; S_T, K, m_{0,T}, T) = (K - m_{0,T})^+ \, . \tag{2.4}$$

Throughout this thesis, S_t will denote the stock-price process at time t. We take an equivalent martingale measure (risk-neutral probability measure) $\mathcal{Q}$ as given (see Duffie, 1996, p.923-924). Under $\mathcal{Q}$, we suppose that the asset price is a time-homogeneous, nonnegative diffusion process solving the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t + \sigma(S_t)\mathrm{d}W_t, \qquad t \geq 0, \qquad S_0 = S > 0, \tag{2.5}$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{Q})$, μ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are the constant risk-free interest rate and the constant dividend yield, respectively), and $\sigma = \sigma(S_t)$ is a given local volatility function, which is assumed to be continuous and strictly positive for all $S \in (0, \infty)$. Note that $\sigma = \sigma(S_t)$ is independent of t.

Using equations (2.1)-(2.4) and risk-neutral valuation, the time-t ($0 \leq t < T$) values of the floating strike lookback call, the floating strike lookback put, the fixed strike lookback call and the fixed strike lookback put are given by:

$$\begin{aligned}
LC_{fl}(t; S_t, m_{0,t}, T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[S_T - m_{0,T} | \mathcal{F}_t\right] , & (2.6)\\
LP_{fl}(t; S_t, M_{0,t}, T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[M_{0,T} - S_T | \mathcal{F}_t\right] , & (2.7)\\
LC_{fx}(t; S_t, K, M_{0,t}, T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T} - K)^+ | \mathcal{F}_t\right] , & (2.8)\\
LP_{fx}(t; S_t, K, m_{0,T}, T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[(K - m_{0,T})^+ | \mathcal{F}_t\right] , & (2.9)
\end{aligned}$$

where all contracts are initiated at time zero, $m_{0,t}$ and $M_{0,t}$ are the minimum and maximum asset prices recorded until date t, S_t is the current underlying asset price at time t, and $\tau = T - t$ is the time remaining to expiration.

To obtain closed-form solutions for the prices of lookback options we need to find the probability distributions of the maximum and minimum asset prices. The next result gives the Laplace transform of the probability distributions of the minimum and maximum asset prices:

Lemma 1. *Let*

$$M_{a,b} := \max_{a \le u \le b} S_u \qquad \text{and} \qquad m_{a,b} := \min_{a \le u \le b} S_u$$

Define $F(y; S, t) := \mathcal{Q}(m_{0,t} \le y | \mathcal{F}_0)$ *and* $G(y; S, t) := \mathcal{Q}(M_{0,t} \ge y | \mathcal{F}_0)$ *(the probabilities are calculated under the risk-neutral measure* $\mathcal{Q}$*). For any* $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} F(y; S, t) \mathrm{d}t = \frac{1}{\lambda} \frac{\phi_\lambda(x)}{\phi_\lambda(y)}, \qquad 0 < y \le S, \tag{2.10}$$

and

$$\int_0^\infty e^{-\lambda t} G(y; S, t) \mathrm{d}t = \frac{1}{\lambda} \frac{\psi_\lambda(x)}{\psi_\lambda(y)}, \qquad 0 < S \le y, \tag{2.11}$$

where ϕ_λ *and* ψ_λ *are the functions defined in Davydov and Linetsky (2001, Proposition 1).*

Proof. If $0 < y \le S$, then

$$\mathcal{Q}\big(m_{0,t} \le y | \mathcal{F}_0\big) = \mathcal{Q}\big(\tau_y \le t | \mathcal{F}_0\big),$$

where $\tau_y := \inf\{t \ge 0 : S_t = y\}$ is the first passage time of the asset process through the level y.

Thus

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} F(y; S, t) \mathrm{d}t &= \int_0^\infty e^{-\lambda t} \mathcal{Q}(m_{0,t} \le y | \mathcal{F}_0) \mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t} \mathcal{Q}(\tau_y \le t | \mathcal{F}_0) \mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t} \mathbb{E}_{\mathcal{Q}} \left[\mathbb{1}_{\{\tau_y \le t\}} \middle| \mathcal{F}_0 \right] \mathrm{d}t \\
&= \lim_{r \to 0^+} \int_0^\infty e^{-\lambda t} \mathbb{E}_{\mathcal{Q}} \big[\mathbb{1}_{\{\tau_y \le t\}} e^{-r \tau_y} | \mathcal{F}_0 \big] \mathrm{d}t \ .
\end{aligned} \tag{2.12}$$

Following Davydov and Linetsky (2001, Proposition 2), then

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_{\mathcal{Q}} \left[\mathbb{1}_{\{\tau_y \le t\}} e^{-r \tau_y} | \mathcal{F}_0 \right] \mathrm{d}t = \frac{1}{\lambda} \mathbb{E}_{\mathcal{Q}} \left[e^{-(r+\lambda) \tau_y} | \mathcal{F}_0 \right] . \tag{2.13}$$

Hence, equations (2.12) and (2.13) yield

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t} F(y;S,t)\mathrm{d}t &= \lim_{r\to 0^+}\left(\frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_y}|\mathcal{F}_0\right]\right) \\
&= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}|\mathcal{F}_0\right] \\
&= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y<\infty\}}|\mathcal{F}_0\right], \qquad (2.14)
\end{aligned}
$$

where the last equality arises because $\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\right]=0$ when $\tau_y=\infty$.

Using Davydov and Linetsky (2001, Equation 2), it is well known that

$$
\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y<\infty\}}|\mathcal{F}_0\right] = \frac{\phi_\lambda(S)}{\phi_\lambda(y)}, \qquad S\ge y\ . \qquad (2.15)
$$

Therefore, combining equations (2.14) and (2.15),

$$
\int_0^\infty e^{-\lambda t}F(y;S,t)\mathrm{d}t = \frac{1}{\lambda}\frac{\phi_\lambda(S)}{\phi_\lambda(y)}, \qquad S\ge y\ .
$$

If $0<S\le y$ and assuming that the barrier at zero is an absorbing barrier, then

$$
\mathcal{Q}(M_{0,t}\ge y|\mathcal{F}_0) = \mathcal{Q}(\tau_y\le t\wedge\tau_y<\tau_0)\ ,
$$

where $\tau_a := \inf\{t\ge 0: S_t=a\}\ ,\ a\in\{0,y\}$.

Thus, if $\tau_y<\tau_0$, then

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}G(y;S,t)\mathrm{d}t &= \int_0^\infty e^{-\lambda t}\mathcal{Q}(M_{0,t}\ge y|\mathcal{F}_0)\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t}\mathcal{Q}(\tau_y\le t|\mathcal{F}_0)\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda t}\mathbb{E}_{\mathcal{Q}}\left[\mathbb{1}_{\{\tau_y\le t\}}|\mathcal{F}_0\right]\mathrm{d}t \\
&= \lim_{r\to 0^+}\int_0^\infty e^{-\lambda t}\mathbb{E}_{\mathcal{Q}}\left[\mathbb{1}_{\{\tau_y\le t\}}e^{-r\tau_y}|\mathcal{F}_0\right]\mathrm{d}t\ .
\end{aligned}
$$

Using again equation (2.13), then

$$\begin{aligned}\int_0^\infty e^{-\lambda t} G(y;S,t)\mathrm{d}t &= \lim_{r\to 0^+}\left(\frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_y}|\mathcal{F}_0\right]\right)\\ &= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}|\mathcal{F}_0\right]\\ &= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y<\tau_0\}}|\mathcal{F}_0\right] + \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y\geq\tau_0\}}|\mathcal{F}_0\right]\\ &= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y<\tau_0\}}|\mathcal{F}_0\right]\,, \end{aligned} \tag{2.16}$$

where the last equality arises because $\{\tau_y \geq \tau_0\} = \emptyset$, since τ_0 represents the default time of the asset.

Using again Davydov and Linetsky (2001, Equation 2), it follows that

$$\mathbb{E}_{\mathcal{Q}}\left[e^{-\lambda\tau_y}\mathbb{1}_{\{\tau_y<\tau_0\}}|\mathcal{F}_0\right] = \frac{\psi_\lambda(S)}{\psi_\lambda(y)}, \qquad S \leq y\,. \tag{2.17}$$

Hence, equations (2.16) and (2.17) can be combined into

$$\int_0^\infty e^{-\lambda t} G(y;S,t)\mathrm{d}t = \frac{1}{\lambda}\frac{\psi_\lambda(S)}{\psi_\lambda(y)}, \qquad S \leq y\,.$$

□

The probability distributions of the minimum and maximum asset prices are recovered by inverting the Laplace transforms. The next proposition provides closed-form solutions to compute the lookback prices in terms of these probabilities.

Proposition 1. *The prices of the floating strike lookback call, the floating strike lookback put, the fixed strike lookback call and the fixed strike lookback put at some time* $0 \leq t < T$ *during the option's life are:*

$$LC_{fl}(t;S_t,m_{0,t},T) = e^{-q\tau}S_t - e^{-r\tau}m_{0,t} + e^{-r\tau}\int_0^{m_{0,t}} F(y;S_t,\tau)\mathrm{d}y \tag{2.18}$$

$$LP_{fl}(t;S_t,M_{0,t},T) = e^{-r\tau}M_{0,t} - e^{-q\tau}S_t + e^{-r\tau}\int_{M_{0,t}}^{\infty} G(y;S_t,\tau)\mathrm{d}y \tag{2.19}$$

$$LC_{fx}(t;S_t,K,M_{0,t},T) = \begin{cases} e^{-r\tau}\int_K^\infty G(y;S_t,\tau)\mathrm{d}y & \Leftarrow M_{0,t}\leq K\\ e^{-r\tau}M_{0,t} - e^{-r\tau}K + e^{-r\tau}\int_{M_{0,t}}^\infty G(y;S_t,\tau)\mathrm{d}y & \Leftarrow M_{0,t} > K\end{cases} \tag{2.20}$$

$$LP_{fx}(t; S_t, K, m_{0,t}, T) = \begin{cases} e^{-r\tau} \int_0^K F(y; S_t, \tau)\mathrm{d}y & \Leftarrow m_{0,t} \geq K \\ e^{-r\tau}K - e^{-r\tau}m_{0,t} + e^{-r\tau} \int_0^{m_{0,t}} F(y; S_t, \tau)\mathrm{d}y & \Leftarrow m_{0,t} < K \end{cases} \tag{2.21}$$

where all contracts are initiated at time zero, $m_{0,t}$ *and* $M_{0,t}$ *are the minimum and maximum prices recorded until date* t*,* S_t *is the current underlying asset price at time* t*,* $\tau = T - t$ *is the time remaining to expiration and* $F(y; S_t, \tau)$ *and* $G(y; S_t, \tau)$ *are defined in Lemma 1.*

Proof. See Appendix A. □

Finally, to compute the price of a lookback option it is only necessary to invert the Laplace transforms offered by equations (2.10) and (2.11), and to plug-in the results into equations (2.18) - (2.21), according to the lookback contract we need to value.

Chapter 3

Turbo Warrants

Let S_t be the underlying asset price at time t. The turbo warrants that we consider can have the following payoffs:

- A **turbo call** warrant pays the option holder $(S_T - K)^+$ at maturity T, where K is the option's strike price, if a prespecified barrier $H \geq K$ has not been hit by S_t at any time before the maturity. If S_t hits the barrier the contract is void and a new contract starts. This new contract is a call option on the minimum process of S_t, with the same strike price K, and time to maturity T_0.

- A **turbo put** warrant pays $(K - S_T)^+$ at maturity T, where K is the option's strike price, if a prespecified barrier $H \leq K$ has not been hit by S_t at any time before the maturity. If S_t hits the barrier the contract is void and a new contract starts. This new contract is a put option on the maximum process of S_t, with the same strike price K, and time to maturity T_0.

From definition above, we can write the value of a turbo call warrant as a sum of two parts. The first part is a down-and-out call option (DOC) with a zero rebate, and the second part is a down and in lookback option (DIL).
Define

$$\tau_H := \inf\{t \geq 0 : S_t \leq H\} \quad \text{and} \quad m_{\tau_H, \tau_H + T_0} := \min_{\tau_H \leq u \leq \tau_H + T_0} S_u \,.$$

By risk-neutral valuation, the value of the DOC and DIL, respectively, is given at time $t < \tau_H$ by the following expressions

$$DOC(t, S) = e^{-r(T-t)} \mathbb{E}_{\mathcal{Q}} \big[(S_T - K)^+ \mathbb{1}_{\{\tau_H > T\}} | \mathcal{F}_t \big] \,, \tag{3.1}$$

and

$$DIL(t, S, T_0) = \mathbb{E}_{\mathcal{Q}} \big[e^{-r(\tau_H + T_0 - t)} (m_{\tau_H, \tau_H + T_0} - K)^+ \mathbb{1}_{\{\tau_H \leq T\}} | \mathcal{F}_t \big] \,. \tag{3.2}$$

The price of the turbo call at time $t < \tau_H$ is given by

$$TC(t,S) = DOC(t,S) + DIL(t,S,T_0)\,. \tag{3.3}$$

The next result gives the price of the turbo call, in closed-form.

Proposition 2. *At $t < \tau_H$, the model-free representation of the turbo call warrant price is given by*

$$TC(t,S) = DOC(t,S) + \mathbb{E}_{\mathcal{Q}}\big[e^{-r(\tau_H - t)}\mathbb{1}_{\{\tau_H \le T\}} LB(\tau_H, S_{\tau_H}, T_0)|\mathcal{F}_t\big]\,, \tag{3.4}$$

where

$$LB(\tau_H, S_{\tau_H}, T_0) = LC_{fl}(\tau_H; S_{\tau_H}, \min\{S_{\tau_H}, K\}, \tau_H + T_0) - LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0). \tag{3.5}$$

Proof. First, we can write $(m_{\tau_H,\tau_H+T_0} - K)^+$ in other form. So,

$$\begin{aligned}
& (m_{\tau_H,\tau_H+T_0} - K)^+ \\
= & \begin{cases} m_{\tau_H,\tau_H+T_0} - K & \Leftarrow m_{\tau_H,\tau_H+T_0} > K \\ 0 & \Leftarrow m_{\tau_H,\tau_H+T_0} \le K \end{cases} \\
= & \begin{cases} (S_{\tau_H+T_0} - K) - (S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}) & \Leftarrow m_{\tau_H,\tau_H+T_0} > K \\ (S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}) - (S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}) & \Leftarrow m_{\tau_H,\tau_H+T_0} \le K \end{cases} \\
= & S_{\tau_H+T_0} - K + (K - m_{\tau_H,\tau_H+T_0})\mathbb{1}_{\{m_{\tau_H,\tau_H+T_0} \le K\}} - (S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}) \\
= & S_{\tau_H+T_0} - K + (K - m_{\tau_H,\tau_H+T_0})^+ - (S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0})\,.
\end{aligned} \tag{3.6}$$

Assuming that $t < \tau_H$, using the tower expectation formula and equation (3.2), we get

$$DIL(t,S,T_0) = \mathbb{E}_{\mathcal{Q}}\Big\{e^{-r(\tau_H - t)}\mathbb{1}_{\{\tau_H \le T\}} LB(\tau_H, S_{\tau_H}, T_0)|\mathcal{F}_t\Big\}\,, \tag{3.7}$$

where

$$LB(\tau_H, S_{\tau_H}, T_0) := \mathbb{E}_{\mathcal{Q}}\big[e^{-rT_0}(m_{\tau_H,\tau_H+T_0} - K)^+|\mathcal{F}_{\tau_H}\big]\,. \tag{3.8}$$

Using equation (3.6),

$$\begin{aligned}
LB(\tau_H, S_{\tau_H}, T_0) &= \mathbb{E}_{\mathcal{Q}}\big[e^{-rT_0}(m_{\tau_H,\tau_H+T_0} - K)^+|\mathcal{F}_{\tau_H}\big] \\
&= \mathbb{E}_{\mathcal{Q}}\Big[e^{-rT_0}\Big\{S_{\tau_H+T_0} - K + (K - m_{\tau_H,\tau_H+T_0})^+ \\
&\quad -(S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0})\Big\}|\mathcal{F}_{\tau_H}\Big] \\
&= e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big] - e^{-rT_0}K + e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[(K - m_{\tau_H,\tau_H+T_0})^+\Big|\mathcal{F}_{\tau_H}\Big] \\
&\quad -e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big] \\
&= e^{-rT_0}S_{\tau_H}e^{(r-q)T_0} - e^{-rT_0}K + e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[(K - m_{\tau_H,\tau_H+T_0})^+\Big|\mathcal{F}_{\tau_H}\Big] \\
&\quad -e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big]\,. \qquad (3.9)
\end{aligned}$$

The last equation was obtained due to the fact that $\mathbb{E}_{\mathcal{Q}}(S_T|\mathcal{F}_t) = S_t e^{(r-q)(T-t)}$.

Comparing equation (2.6) with $e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big]$, we conclude that this expression is simply the value at time τ_H of a floating strike lookback call, on asset S, inception at time τ_H and expiry date at time $\tau_H + T_0$. Thus,

$$e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big] = LC_{fl}(\tau_H; S_{\tau_H}, m_{\tau_H,\tau_H}, \tau_H + T_0)\,.$$

Notice that $m_{\tau_H,\tau_H} = \min_{\{\tau_H \le u \le \tau_H\}} S_u = S_{\tau_H}$. Therefore,

$$e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[S_{\tau_H+T_0} - m_{\tau_H,\tau_H+T_0}\Big|\mathcal{F}_{\tau_H}\Big] = LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0)\,. \qquad (3.10)$$

Comparing expression $e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[(K - m_{\tau_H,\tau_H+T_0})^+\Big|\mathcal{F}_{\tau_H}\Big]$ with equation (2.9), we conclude that this expression is the value at time τ_H of a fixed strike lookback put, on asset S, strike price equal to K, inception at time τ_H and expiry date at time $\tau_H + T_0$. Thus,

$$\begin{aligned}
e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[(K - m_{\tau_H,\tau_H+T_0})^+\Big|\mathcal{F}_{\tau_H}\Big] &= LP_{fx}(\tau_H; S_{\tau_H}, K, m_{\tau_H,\tau_H}, \tau_H + T_0) \\
&= LP_{fx}(\tau_H; S_{\tau_H}, K, S_{\tau_H}, \tau_H + T_0)\,. \qquad (3.11)
\end{aligned}$$

Using equation (2.21), the last equation becomes

$$\begin{aligned}
&e^{-rT_0}\mathbb{E}_{\mathcal{Q}}\Big[(K - m_{\tau_H,\tau_H+T_0})^+\Big|\mathcal{F}_{\tau_H}\Big] \\
&= \begin{cases} e^{-rT_0}\int_0^K F(y; S_{\tau_H}, T_0)\mathrm{d}y & \Leftarrow S_{\tau_H} \ge K \\ e^{-rT_0}K - e^{-rT_0}S_{\tau_H} + e^{-rT_0}\int_0^{S_{\tau_H}} F(y; S_{\tau_H}, T_0)\mathrm{d}y & \Leftarrow S_{\tau_H} < K \end{cases}, \qquad (3.12)
\end{aligned}$$

where $F(y; S_{\tau_H}, T_0) = \mathcal{Q}(m_{\tau_H,\tau_H+T_0} \le y|\mathcal{F}_{\tau_H})$.

Combining equations (3.9), (3.10) and (3.12), we get

$$
\begin{aligned}
& LB(\tau_H, S_{\tau_H}, T_0) \\
= & \begin{cases} e^{-qT_0} S_{\tau_H} - e^{-rT_0} K + e^{-rT_0} \int_0^K F(y; S_{\tau_H}, T_0) \mathrm{d}y \\ \quad -LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0) & \Leftarrow S_{\tau_H} \geq K \\ \\ e^{-qT_0} S_{\tau_H} - e^{-rT_0} K + e^{-rT_0} K - e^{-rT_0} S_{\tau_H} \\ \quad + e^{-rT_0} \int_0^{S_{\tau_H}} F(y; S_{\tau_H}, T_0) \mathrm{d}y - LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0) & \Leftarrow S_{\tau_H} < K \end{cases} \\
= & \begin{cases} e^{-qT_0} S_{\tau_H} - e^{-rT_0} K + e^{-rT_0} \int_0^K F(y; S_{\tau_H}, T_0) \mathrm{d}y \\ \quad -LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0) & \Leftarrow S_{\tau_H} \geq K \\ \\ e^{-qT_0} S_{\tau_H} - e^{-rT_0} S_{\tau_H} + e^{-rT_0} \int_0^{S_{\tau_H}} F(y; S_{\tau_H}, T_0) \mathrm{d}y \\ \quad -LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0) & \Leftarrow S_{\tau_H} < K \end{cases} \\
= & \; e^{-qT_0} S_{\tau_H} - e^{-rT_0} \min\{S_{\tau_H}, K\} + e^{-rT_0} \int_0^{\min\{S_{\tau_H}, K\}} F(y; S_{\tau_H}, T_0) \mathrm{d}y \\
& \quad -LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0)\,.
\end{aligned}
$$

Using equation (2.18), the last equation becomes

$$
LB(\tau_H, S_{\tau_H}, T_0) = LC_{fl}(\tau_H; S_{\tau_H}, \min\{S_{\tau_H}, K\}, \tau_H + T_0) - LC_{fl}(\tau_H; S_{\tau_H}, S_{\tau_H}, \tau_H + T_0)\,. \tag{3.13}
$$

Equations (3.3), (3.7) and (3.13) yield equation (3.4). □

If the underlying asset price follows a stochastic process of continuous sample paths, then we have $S_{\tau_H} = H$. So the next corollary simplifies the price of a turbo call warrant under this condition.

Corollary 1. *If the asset price follows a continuous diffusion process, then, at $t < \tau_H$, the turbo call price reads*

$$
TC(t, S) = DOC(t, S) + \mathbb{E}_{\mathcal{Q}}\big[e^{-r(\tau_H - t)} \mathbb{1}_{\{\tau_H \leq T\}} LB(\tau_H, H, T_0) | \mathcal{F}_t\big]\,, \tag{3.14}
$$

where $LB(\tau_H, H, T_0)$ is obtained by replacing S_{τ_H} for H in equation (3.5).

Proof. Equation (3.14) follows from Proposition 2 because if the asset price follows a continuous diffusion process, then $S_{\tau_H} = H$. □

Corollary 1 can be applied to local volatility and stochastic volatility models. If the asset price is a diffusion process like in equation (2.5) then the following result holds. Note that the Black-Scholes and the CEV models are special cases of the time-independent local volatility model.

Corollary 2. *If the asset price follows a time-independent local volatility model, then, at $t < \tau_H$, the turbo call price reads*

$$TC(t,S) = DOC(t,S) + LB(0,H,T_0)\mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_H - t)}\mathbb{1}_{\{\tau_H \leq T\}}|\mathcal{F}_t\right], \tag{3.15}$$

where

$$LB(0,H,T_0) = LC_{fl}(0;H,K,T_0) - LC_{fl}(0;H,H,T_0). \tag{3.16}$$

Proof. Replacing S_{τ_H} for H in equation (3.13), we get

$$LB(\tau_H,H,T_0) = LC_{fl}(\tau_H;H,\min\{H,K\},\tau_H + T_0) - LC_{fl}(\tau_H;H,H,\tau_H + T_0).$$

According to the definition of a turbo warrant call, we know that $H \geq K$. So, the last equation becomes

$$LB(\tau_H,H,T_0) = LC_{fl}(\tau_H;H,K,\tau_H + T_0) - LC_{fl}(\tau_H;H,H,\tau_H + T_0). \tag{3.17}$$

Using equation (2.18), the last equation can be written as

$$\begin{aligned} LB(\tau_H,H,T_0) \quad = \quad & e^{-qT_0}H - e^{-rT_0}K + e^{-rT_0}\int_0^K F(y;H,T_0)\mathrm{d}y \\ & -e^{-qT_0}H + e^{-rT_0}H - e^{-rT_0}\int_0^H F(y;H,T_0)\mathrm{d}y. \end{aligned}$$

Simplifying, the last equation becomes

$$LB(\tau_H,H,T_0) = (H-K)e^{-rT_0} - e^{-rT_0}\int_K^H F(y;H,T_0)\mathrm{d}y. \tag{3.18}$$

The function $F(y;H,T_0)$ is equal to $\mathcal{Q}(m_{\tau_H,\tau_H+T_0} \leq y|\mathcal{F}_{\tau_H})$, but can also be written as $\mathcal{Q}(m_{0,T_0} \leq y|\mathcal{F}_0)$ given by Markovian nature of the pricing model under analysis, which means that the only relevant information on the future of the process is its present value, and its past is irrelevant. Therefore, equation (3.18) can be written as

$$LB(\tau_H,H,T_0) = (H-K)e^{-rT_0} - e^{-rT_0}\int_K^H \mathcal{Q}(m_{0,T_0} \leq y|\mathcal{F}_0)\mathrm{d}y,$$

i.e.,

$$LB(\tau_H,H,T_0) = LB(0,H,T_0). \tag{3.19}$$

which is independent of τ_H. Replacing τ_H by 0 in equation (3.17) we get equation (3.16).

Combing equations (3.14) and (3.19), the turbo call price reads

$$
\begin{aligned}
TC(t,S) &= DOC(t,S) + \mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_H - t)}\mathbb{1}_{\{\tau_H \le T\}} LB(0,H,T_0)|\mathcal{F}_t\right] \\
&= DOC(t,S) + LB(0,H,T_0)\mathbb{E}_{\mathcal{Q}}\left[e^{-r(\tau_H - t)}\mathbb{1}_{\{\tau_H \le t\}}|\mathcal{F}_t\right].
\end{aligned}
$$

The last equation arises because $LB(0,H,T_0)$ is $\mathcal{F}_t$ - measurable. □

To compute the price of the turbo warrant under a time-independent local volatility model, we need to compute the DOC option, as well as the LB part and the expectation $\mathbb{E}_{\mathcal{Q}}[e^{-r(\tau_H-t)}\mathbb{1}_{\{\tau_H \le t\}}|\mathcal{F}_t]$. If the asset price is a diffusion process solving the stochastic differential equation (2.5) and respecting all the assumptions that were referred in Chapter 2, then we can follow Davydov and Linetsky (2001) to compute both parts:

- The DOC component in the turbo call warrant is given by equation (3.1). Let $\tau_{(H,U)} := \inf\{t \ge 0 : S_t \notin (H,U)\}$. Then $\lim_{U\to\infty} \tau_{(H,U)} = \tau_H$. Let

$$DBKO(t,S) = e^{-r(T-t)}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(H,U)}>T\}}(S_T - K)^+|\mathcal{F}_t] \tag{3.20}$$

  be the value of a double barrier knock out option at time t. To compute the DOC option we need to take the limit of DBKO as U approaches infinity.

  Davydov and Linetsky (2001) have obtained a closed-form solution for double barrier options under a general one-dimensional diffusion, when the strike price is between the lower and the upper barriers. To compute the DOC component in the turbo call we have to modify their solution because the strike price is less than the downside barrier and we have to take the limit $U \to \infty$. Next result presents the expected value of the terminal payoff of a double knock out option in terms of its Laplace Transform, when the strike price is less than the downside barrier.

  Proposition 3. *The Laplace Transform of the expected value of the terminal payoff of a double knock out option, when the strike price (K) is less than the downside barrier (L) is given by:*

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(L,U)}>T\}}(S_T - K)^+]\mathrm{d}T \\
&= \frac{1}{\omega_\lambda \Delta_\lambda(L,U)}\Big(\Delta_\lambda(L,S)\left[\psi_\lambda(U)J_\lambda(K,S,U) - \phi_\lambda(U)I_\lambda(K,S,U)\right] \\
&\quad + \Delta_\lambda(S,U)\left[\phi_\lambda(L)I_\lambda(K,L,S) - \psi_\lambda(L)J_\lambda(K,L,S)\right]\Big),
\end{aligned} \tag{3.21}
$$

where ω_λ is defined in Davydov and Linetsky (2001, Equation 13) through the ODE

$$\phi_\lambda(S)\frac{\mathrm{d}\psi_\lambda}{\mathrm{d}S}(S) - \psi_\lambda(S)\frac{\mathrm{d}\phi_\lambda}{\mathrm{d}S}(S) = \mathfrak{s}(S)w_\lambda\,, \tag{3.22}$$

and

$$\Delta_\lambda(A,B) := \phi_\lambda(A)\psi_\lambda(B) - \psi_\lambda(A)\phi_\lambda(B)\,. \tag{3.23}$$

Moreover,

$$I_\lambda(K,A,B) = \int_A^B (Y-K)\psi_\lambda(Y)m(Y)\mathrm{d}Y, \tag{3.24}$$

and

$$J_\lambda(K,A,B) = \int_A^B (Y-K)\phi_\lambda(Y)m(Y)\mathrm{d}Y, \tag{3.25}$$

where $m(Y)$ is the speed density of the diffusion (2.5) *that is defined by*

$$m(Y) := \frac{2}{\sigma^2(Y)Y^2\mathfrak{s}(Y)}. \tag{3.26}$$

In equations (3.22) *and* (3.26)*, $\mathfrak{s}(S)$ is the scale density of diffusion* (2.5)*, which is defined as*

$$\mathfrak{s}(S) := \exp\left\{-\int\frac{2\mu}{\sigma^2(S)S}\mathrm{d}S\right\}. \tag{3.27}$$

Proof. See Appendix B. □

Therefore, to compute the *DOC* option we need to take the limit $U \to \infty$ in equation (3.20) and use the boundary properties of the functions ϕ_λ and ψ_λ defined in Davydov and Linetsky (2001, Proposition 1). The useful boundary properties are

$$\lim_{S\to\infty}\psi_\lambda(S) = +\infty \qquad \lim_{S\to\infty}\phi_\lambda(S) = 0\,. \tag{3.28}$$

Under certain conditions, we can get a simple expression of the *DOC* option. Next result provides this expression.

Proposition 4. *If*

$$\lim_{U\to\infty} J_\lambda(K,S,U) < +\infty \quad \text{and} \quad \lim_{U\to\infty} I_\lambda(K,S,U) < +\infty\,, \tag{3.29}$$

or if

$$\lim_{U\to\infty} J_\lambda(K,S,U) < +\infty \quad \text{and} \quad \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) = 0 \tag{3.30}$$

then the Laplace transform of the expected value of the terminal payoff of a down-and-out call option, when the strike price (K) is less than the downside barrier (L), is given by

$$\begin{aligned}
&\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_L>T\}}(S_T-K)^+]\mathrm{d}T \\
= \ & \frac{1}{\omega_\lambda \phi_\lambda(L)}\Big(\Delta_\lambda(L,S)\lim_{U\to\infty}\big(J_\lambda(K,S,U)\big) + \phi_\lambda(S)\big[\phi_\lambda(L)I_\lambda(K,L,S) \\
&-\psi_\lambda(L)J_\lambda(K,L,S)\big]\Big), \qquad (3.31)
\end{aligned}$$

where $\tau_L := \inf\{t \geq 0 : S_t = L\}$.

Proof. Using equations (3.21) and (3.23), we get

$$\begin{aligned}
&\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_L>T\}}(S_T-K)^+]\mathrm{d}T \\
= \ & \lim_{U\to\infty}\Bigg\{\frac{1}{\omega_\lambda(\phi_\lambda(L)\psi_\lambda(U)-\psi_\lambda(L)\phi_\lambda(U))} \\
&\Big(\Delta_\lambda(L,S)\big[\psi_\lambda(U)J_\lambda(K,S,U)-\phi_\lambda(U)I_\lambda(K,S,U)\big] \\
&+\big(\phi_\lambda(S)\psi_\lambda(U)-\psi_\lambda(S)\phi_\lambda(U)\big)\big[\phi_\lambda(L)I_\lambda(K,L,S)-\psi_\lambda(L)J_\lambda(K,L,S)\big]\Big)\Bigg\} \\
= \ & \lim_{U\to\infty}\Bigg\{\frac{1}{\omega_\lambda\big(\phi_\lambda(L)-\psi_\lambda(L)\phi_\lambda(U)\frac{1}{\psi_\lambda(U)}\big)} \\
&\Big(\Delta_\lambda(L,S)\big[J_\lambda(K,S,U)-\phi_\lambda(U)\frac{1}{\psi_\lambda(U)}I_\lambda(K,S,U)\big] \\
&+\left(\phi_\lambda(S)-\psi_\lambda(S)\phi_\lambda(U)\frac{1}{\psi_\lambda(U)}\right)\big[\phi_\lambda(L)I_\lambda(K,L,S)-\psi_\lambda(L)J_\lambda(K,L,S)\big]\Big)\Bigg\} \\
= \ & \frac{1}{\omega_\lambda \phi_\lambda(L)}\Big(\Delta_\lambda(L,S)\lim_{U\to\infty}\big(J_\lambda(K,S,U)\big) + \phi_\lambda(S)\big[\phi_\lambda(L)I_\lambda(K,L,S) \\
&-\psi_\lambda(L)J_\lambda(K,L,S)\big]\Big).
\end{aligned}$$

□

Therefore, to find the price of the *DOC* option we need to invert the Laplace transform of equation (3.31) and multiply the result by $e^{-r(T-t)}$.

- The $LB(0, H, T_0)$ formula can be obtained using the following equation

$$LB(0, H, T_0) = (H - K)e^{-rT_0} - e^{-rT_0} \int_K^H F(y; H, T_0)\mathrm{d}y \,, \tag{3.32}$$

where $F(y; H, T_0)$ is obtained by inverting the Laplace transform of equation (2.10).

- The expectation $\mathbb{E}_{\mathcal{Q}}[e^{-r(\tau_H - t)} \mathbb{1}_{\{\tau_H \leq T\}} | \mathcal{F}_t]$ can be computed using two results of Davydov and Linetsky (2001). Let

$$DR(t, S) = \mathbb{E}_{\mathcal{Q}}[e^{-r(\tau_H - t)} \mathbb{1}_{\{\tau_H \leq T\}} | \mathcal{F}_t] \,. \tag{3.33}$$

Using the fact that the asset price is a time-homogeneous diffusion process, then we can write this in another way:

$$DR(t, S) = \mathbb{E}_{\mathcal{Q}}\left[e^{-r\tau_H} \mathbb{1}_{\{\tau_H \leq T\}} | \mathcal{F}_0\right]$$

Using the result of Davydov and Linetsky (2001, Proposition 2),

$$\begin{aligned} \int_0^\infty e^{-\lambda t} DR(t, S)\mathrm{d}t &= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_H} | \mathcal{F}_0\right] \\ &= \frac{1}{\lambda}\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_H} \mathbb{1}_{\{\tau_H < \infty\}} | \mathcal{F}_0\right] \,, \end{aligned} \tag{3.34}$$

because $\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_H}\right] = 0$ when $\tau_H = \infty$. Using Davydov and Linetsky (2001, Equation 2), we get

$$\mathbb{E}_{\mathcal{Q}}\left[e^{-(r+\lambda)\tau_H} \mathbb{1}_{\{\tau_H < \infty\}} | \mathcal{F}_0\right] = \frac{\phi_{r+\lambda}(S)}{\phi_{r+\lambda}(H)} \,. \tag{3.35}$$

Combining equations (3.34) and (3.35), we get the Laplace transform of DR:

$$\int_0^\infty e^{-\lambda t} DR(t, S)\mathrm{d}t = \frac{1}{\lambda}\frac{\phi_{r+\lambda}(S)}{\phi_{r+\lambda}(H)} \,. \tag{3.36}$$

Therefore, the DR value is obtained by inverting the Laplace transform of the last equation.

Chapter 4

GBM Model

In 1973, Black and Scholes (1973) and Merton (1973) published their papers on the theory of option pricing. They developed the Black-Scholes model, which has had a huge influence on the development of option and derivative industries.

The Black-Scholes risk-neutral formulation of the option pricing theory is attractive due to the fact that the pricing formula of a derivative is a function of directly observable parameters (with exception of the volatility parameter). The assumptions of this model are:

i) The underlying asset price follows a geometric Brownian (GBM) motion:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t^Q\,,$$

with r and σ constant.

ii) The short selling of securities with full use of proceeds is permitted.

iii) There are no transactions costs or taxes.

iv) The assets are perfectly divisible.

v) The asset pays no dividend.

vi) There are no riskless arbitrage opportunities.

vii) Trading takes place continuously in time.

viii) The risk-free rate of interest, r, is constant and the same for all maturities.

The Black and Scholes (1973) option pricing model can be extended to deal with options on dividend-paying stocks:

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t^Q, \tag{4.1}$$

where $\mathcal{Q}$ is an equivalent martingale measure (risk-neutral probability measure). Under $\mathcal{Q}$, we suppose that the asset price is a time-homogeneous, nonnegative diffusion process; $\{W_t^Q : t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{Q})$ and μ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are the constant risk-free interest rate and the constant dividend yield, respectively). Under this model, the stock return $\frac{\mathrm{d}S_t}{S_t}$ in the small interval $\mathrm{d}t$ is normally distributed with mean $\mu \mathrm{d}t$ and variance $\sigma^2 \mathrm{d}t$.

Unfortunately, this model is not consistent with the empirical evidence, as explained in Chapter 1. In the next chapter we will study the CEV model, wherein the prices are closer to market quotes.

4.1 Lookback options under the GBM Model

We can compute the price of the lookback options under the Black-Scholes model in two different ways: using equations (2.10) - (2.11) and (2.18) - (2.21), or using the pricing formulae of Zhang (1998, p.341-352).

To compute the price of lookback options using equations (2.10) - (2.11) and (2.18) - (2.21), functions ψ_λ and ϕ_λ can be replaced in equations (2.10) and (2.11) by closed-form solutions. Functions ψ_λ and ϕ_λ are defined in Davydov and Linetsky (2001, equation 15):

$$\psi_\lambda = S^{\gamma_+}, \tag{4.2}$$

and

$$\phi_\lambda = S^{\gamma_-}, \tag{4.3}$$

where

$$\gamma_\pm = -\gamma \pm \sqrt{\gamma^2 + \frac{2\lambda}{\sigma^2}}, \tag{4.4}$$

with

$$\gamma = \frac{\mu}{\sigma^2} - \frac{1}{2}. \tag{4.5}$$

Using the pricing formulae of Zhang (1998, p. 346-352),[1] the prices of the floating strike lookback call, the floating strike lookback put, the fixed strike lookback call and the

[1] Equations (4.6)-(4.9) are different from Zhang (1998) when $r = q$, i.e. when the *drift* is zero. The proof of these equations for the case $r = q$ is given in Appendix C.

fixed strike lookback put at some time $0 \le t < T$ and during the option's life are:

$$
\begin{aligned}
&LC_{fl}(t; S_t, m_{0,t}, T) \\
&= \begin{cases}
S_t e^{-q\tau}\Phi\big[d_{bs1}(S_t, m_{0,t})\big] - m_{0,t}e^{-r\tau}\Phi\big[d_{bs}(S_t, m_{0,t})\big] \\
\quad +\frac{S_t}{\delta}\Big\{e^{-r\tau}\Big(\frac{S_t}{m_{0,t}}\Big)^{-\delta}\Phi\big[d_{bs}(m_{0,t}, S_t)\big] - e^{-q\tau}\Phi\big[-d_{bs1}(S_t, m_{0,t})\big]\Big\}, & \Leftarrow r \neq q \\
S_t e^{-q\tau}\Phi\big[d_{bs1}(S_t, m_{0,t})\big] - m_{0,t}e^{-r\tau}\Phi\big[d_{bs}(S_t, m_{0,t})\big] \\
\quad +S_t e^{-r\tau}\sigma\sqrt{\tau} f\big[-d_{bs1}(S_t, m_{0,t})\big] \\
\quad -S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi\big[-d_{bs1}(S_t, m_{0,t})\big]\Big\{\frac{2}{\sigma^2}\ln\Big(\frac{S_t}{m_{0,t}}\Big) + \tau\Big\}, & \Leftarrow r = q
\end{cases}
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&LP_{fl}(t; S_t, M_{0,t}, T) \\
&= \begin{cases}
M_{0,t}e^{-r\tau}\Phi\big[-d_{bs}(S_t, M_{0,t})\big] - S_t e^{-q\tau}\Phi\big[-d_{bs1}(S_t, M_{0,t})\big] \\
\quad +\frac{S_t}{\delta}\Big\{e^{-q\tau}\Phi\big[d_{bs1}(S_t, M_{0,t})\big] - e^{-r\tau}\Big(\frac{S_t}{M_{0,t}}\Big)^{-\delta}\Phi\big[-d_{bs}(M_{0,t}, S_t)\big]\Big\}, & \Leftarrow r \neq q \\
M_{0,t}e^{-r\tau}\Phi\big[-d_{bs}(S_t, M_{0,t})\big] - S_t e^{-q\tau}\Phi\big[-d_{bs1}(S_t, M_{0,t})\big] \\
\quad +S_t e^{-r\tau}\sigma\sqrt{\tau} f\big[d_{bs1}(S_t, M_{0,t})\big] \\
\quad +S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi\big[d_{bs1}(S_t, M_{0,t})\big]\Big\{\frac{2}{\sigma^2}\ln\Big(\frac{S_t}{M_{0,t}}\Big) + \tau\Big\}, & \Leftarrow r = q
\end{cases}
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&LC_{fx}(t; S_t, K, M_{0,t}, T) \\
&= \begin{cases}
e^{-r\tau}(M_{0,t} - K)^+ + S_t e^{-q\tau}\Phi\big[d_{bs1}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad -\max\{M_{0,t}, K\}e^{-r\tau}\Phi\big[d_{bs}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad +\frac{S_t}{\delta}\Big\{e^{-q\tau}\Phi\big[d_{bs1}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad -e^{-r\tau}\Big(\frac{S_t}{\max\{M_{0,t},K\}}\Big)^{-\delta}\Phi\big[-d_{bs}(\max\{M_{0,t}, K\}, S_t)\big]\Big\}, & \Leftarrow r \neq q \\
e^{-r\tau}(M_{0,t} - K)^+ + S_t e^{-q\tau}\Phi\big[d_{bs1}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad -\max\{M_{0,t}, K\}e^{-r\tau}\Phi\big[d_{bs}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad +S_t e^{-r\tau}\sigma\sqrt{\tau} f\big[d_{bs1}(S_t, \max\{M_{0,t}, K\})\big] \\
\quad +S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi\big[d_{bs1}(S_t, \max\{M_{0,t}, K\})\big]\Big\{\frac{2}{\sigma^2}\ln\Big(\frac{S_t}{\max\{M_{0,t},K\}}\Big) + \tau\Big\}, & \Leftarrow r = q
\end{cases}
\end{aligned}
\tag{4.8}
$$

$$
LP_{fx}(t;S_t,K,m_{0,t},T)
$$

$$
= \begin{cases} e^{-r\tau}(K-m_{0,t})^+ + \min\{m_{0,t},K\}e^{-r\tau}\Phi\big[-d_{bs}(S_t,\min\{m_{0,t},K\})\big] \\ \quad -S_t e^{-q\tau}\Phi\big[-d_{bs1}(S_t,\min\{m_{0,t},K\})\big] \\ \quad +\frac{S_t}{\delta}\Big\{-e^{-q\tau}\Phi\big[-d_{bs1}(S_t,\min\{m_{0,t},K\})\big] \\ \quad +e^{-r\tau}\Big(\frac{S_t}{\min\{m_{0,t},K\}}\Big)^{-\delta}\Phi\big[d_{bs}(\min\{m_{0,t},K\},S_t)\big]\Big\}, & \Leftarrow r\neq q \\ e^{-r\tau}(K-m_{0,t})^+ + \min\{m_{0,t},K\}e^{-r\tau}\Phi\big[-d_{bs}(S_t,\min\{m_{0,t},K\})\big] \\ \quad -S_t e^{-q\tau}\Phi\big[-d_{bs1}(S_t,\min\{m_{0,t},K\})\big] \\ \quad +S_t e^{-r\tau}\sigma\sqrt{\tau}f\big[-d_{bs1}(S_t,\min\{m_{0,t},K\})\big] \\ \quad -S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi\big[-d_{bs1}(S_t,\min\{m_{0,t},K\})\big]\Big\{\frac{2}{\sigma^2}\ln\Big(\frac{S_t}{\min\{m_{0,t},K\}}\Big)+\tau\Big\}, & \Leftarrow r=q \end{cases} \tag{4.9}
$$

where $\Phi[.]$ is the normal cumulative distribution function, $f[.]$ is the normal probability density function,

$$
d_{bs}(Y,X) = \frac{\ln(Y/X)+(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \tag{4.10}
$$

$$
d_{bs1}(Y,X) = d_{bs}(Y,X)+\sigma\sqrt{\tau}, \tag{4.11}
$$

$$
\delta = \frac{2(r-q)}{\sigma^2}, \tag{4.12}
$$

and where all contracts are initiated at time zero, $m_{0,t}$ and $M_{0,t}$ are the minimum and maximum prices recorded until date t, S_t is the current underlying asset price at time t and $\tau = T-t$ is the time remaining to expiration.

4.2 Turbo Warrants under the GBM Model

The explicit solution of turbo warrants under the Black-Scholes model can be obtained by replacing functions ψ_λ and ϕ_λ given by equations (4.2) and (4.3), repectively, into equations (3.31),[2] (3.32) and (3.36). For the *DOC* component we have to find the expression of speed density of diffusion (4.1) to compute in closed form the integrals (3.24) and (3.25).

Then, the scale density is defined by (3.27). Replacing $\sigma(S)$ for σ, the scale density

[2]See Proposition 5 in page 26 for the proof of conditions of Proposition 4.

becomes

$$\begin{aligned}\mathfrak{s}(S) &= \exp\left\{-\frac{2\mu}{\sigma^2}\int\frac{1}{S}\mathrm{d}S\right\} \\ &= \exp\left\{\ln\left(S^{-\frac{2\mu}{\sigma^2}}\right)\right\} \\ &= S^{-\frac{2\mu}{\sigma^2}}\,. \end{aligned} \tag{4.13}$$

Therefore, the speed density defined by equation (3.26) becomes

$$\begin{aligned} m(S) &= \frac{2}{\sigma^2 S^2 S^{-\frac{2\mu}{\sigma^2}}} \\ &= \frac{2}{\sigma^2 S^{2-\frac{2\mu}{\sigma^2}}}\,. \end{aligned} \tag{4.14}$$

Replacing the functions ψ_λ and ϕ_λ given by equations (4.2) and (4.3) and the speed density in equation (4.14) in equation (3.22), the Wronskian of the functions ψ_λ and ϕ_λ with respect of the scale density of equation (4.13) is given by

$$\begin{aligned} S^{\gamma_-}S^{\gamma_+-1}\gamma_+ - S^{\gamma_+}S^{\gamma_--1}\gamma_- &= S^{-\frac{2\mu}{\sigma^2}}\omega_\lambda \\ S^{-2\gamma-1}\left(2\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right) &= S^{-\frac{2\mu}{\sigma^2}}\omega_\lambda\,. \end{aligned} \tag{4.15}$$

The last equation arises because $\gamma_\pm = -\gamma \pm \sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}$. Using the fact that $\gamma = \frac{\mu}{\sigma^2}-\frac{1}{2}$, equation (4.15) becomes

$$\begin{aligned} 2S^{-2\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)-1}\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}} &= S^{-\frac{2\mu}{\sigma^2}}\omega_\lambda \\ 2\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}} = \omega_\lambda\,. \end{aligned} \tag{4.16}$$

The integrals I_λ and J_λ are defined in equations (3.24) and (3.25), respectively. Replacing the functions ψ_λ and ϕ_λ given by equations (4.2)-(4.3) and the speed density given by equation (4.14) into equations (3.24) and (3.25), and using the definitions stated in equations (4.4) and (4.5), these integrals become:

$$
\begin{aligned}
& I_\lambda(K,A,B) \\
= \;& \int_A^B (Y-K)Y^{\gamma_+}\frac{2}{\sigma^2 Y^{2-\frac{2\mu}{\sigma^2}}}\mathrm{d}Y \\
= \;& \frac{2}{\sigma^2}\left(\int_A^B Y^{\gamma_+-1+\frac{2\mu}{\sigma^2}}\mathrm{d}Y - K\int_A^B Y^{\gamma_+-2+\frac{2\mu}{\sigma^2}}dY\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{\gamma_++\frac{2\mu}{\sigma^2}}}{\gamma_++\frac{2\mu}{\sigma^2}}\right]_A^B - K\left[\frac{Y^{\gamma_++\frac{2\mu}{\sigma^2}-1}}{\gamma_++\frac{2\mu}{\sigma^2}-1}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}}{\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}\right]_A^B - K\left[\frac{Y^{\frac{\mu}{\sigma^2}-\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}}{\frac{\mu}{\sigma^2}-\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{1-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}}{1-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}\right]_A^B - K\left[\frac{Y^{-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}}{-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{1-\gamma_-}}{1-\gamma_-}\right]_A^B + K\left[\frac{Y^{-\gamma_-}}{\gamma_-}\right]_A^B\right)
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
& J_\lambda(K,A,B) \\
= \;& \int_A^B (Y-K)Y^{\gamma_-}\frac{2}{\sigma^2 Y^{2-\frac{2\mu}{\sigma^2}}}\mathrm{d}Y \\
= \;& \frac{2}{\sigma^2}\left(\int_A^B Y^{\gamma_--1+\frac{2\mu}{\sigma^2}}\mathrm{d}Y - K\int_A^B Y^{\gamma_--2+\frac{2\mu}{\sigma^2}}dY\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{\gamma_-+\frac{2\mu}{\sigma^2}}}{\gamma_-+\frac{2\mu}{\sigma^2}}\right]_A^B - K\left[\frac{Y^{\gamma_-+\frac{2\mu}{\sigma^2}-1}}{\gamma_-+\frac{2\mu}{\sigma^2}-1}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}}{\frac{\mu}{\sigma^2}+\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}\right]_A^B - K\left[\frac{Y^{\frac{\mu}{\sigma^2}-\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}}{\frac{\mu}{\sigma^2}-\frac{1}{2}-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{1-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}}{1-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}\right]_A^B - K\left[\frac{Y^{-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}}{-\left(-\frac{\mu}{\sigma^2}+\frac{1}{2}+\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}\right)}\right]_A^B\right) \\
= \;& \frac{2}{\sigma^2}\left(\left[\frac{Y^{1-\gamma_+}}{1-\gamma_+}\right]_A^B + K\left[\frac{Y^{-\gamma_+}}{\gamma_+}\right]_A^B\right).
\end{aligned}
\tag{4.18}
$$

To apply the equation (3.31) we need to prove that the conditions in equation (3.30) are true. Next proposition give us that proof and the closed-form for $\lim_{U\to\infty} J_\lambda(K,S,U)$.

Proposition 5. *Under GBM assumptions, for any $S>0$ and for $\lambda>0$ large such that*

$$1+\gamma-\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}<0, \tag{4.19}$$

we have

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)=0\,, \tag{4.20}$$

and

$$\lim_{U\to\infty}J_\lambda(K,S,U)=-\frac{2}{\sigma^2}\left(\frac{S^{1-\gamma_+}}{1-\gamma_+}+K\frac{S^{-\gamma_+}}{\gamma_+}\right), \tag{4.21}$$

where γ_+ and γ_- are defined in equation (4.4) and γ is defined in equation (4.5).

Proof. See Appendix D. □

Therefore, to compute the DOC component of the turbo call we need to replace ψ_λ, ϕ_λ, ω_λ, I_λ, J_λ and $\lim_{U\to\infty} J_\lambda(K,S,U)$, defined (for this model) in equations (4.2), (4.3), (4.16), (4.17), (4.18) and (4.21), respectively, into equation (3.31), using the definition of $\Delta(A,B)$ given in equation (3.23). After that, we have to invert the Laplace transform of equation (3.31) and multiply the result by $e^{-r(T-t)}$.

For the LB component we have to invert the Laplace transform of equation (2.10), replacing ϕ_λ defined in equation (4.3) into equation (2.10) to compute $F(y;H,T_0)$. The result must be replace in equation (3.32).

The DR value is obtained by inverting the Laplace transform of equation (3.36), replacing ϕ_λ defined in equation (4.3) into equation (3.36).

After computing the values of DOC, LB and DR, we have to replace them into equation (3.15).

Another way to compute the Black-Scholes pricing formulas of the turbo call warrant is combining equations (3.15)-(3.16) and (3.33):

$$TC_{BS}(t,S)=DOC(t,S)+\big(LC_{fl}(0;H,K,T_0)-LC_{fl}(0;H,H,T_0)\big)DR(t,S)\,, \tag{4.22}$$

where the formula for the floating strike lookback call is given by equation (4.6) and the pricing formulae of the DOC and DR components are obtained following Zhang (1998, p.233 and p.240):

$$
\begin{aligned}
DOC(t,S) \;=\; & S_t e^{-q\tau}\Phi\big[d_{bs1}(S_t,\max\{H,K\})\big] - \max\{H,K\}e^{-r\tau}\Phi\big[d_{bs}(S_t,\max\{H,K\})\big] \\
& -\left(\frac{H}{S_t}\right)^{\delta-1}\left\{\frac{H^2}{S_t}e^{-q\tau}\Phi\left[d_{bs1}\left(\frac{H^2}{S_t},\max\{H,K\}\right)\right]\right. \\
& \left.-\max\{H,K\}e^{-r\tau}\Phi\left[d_{bs}\left(\frac{H^2}{S_t},\max\{H,K\}\right)\right]\right\} \\
& +\left(\max\{H,K\}-K\right)e^{-r\tau}\left\{\Phi\big[d_{bs}(S_t,\max\{H,K\})\big]\right. \\
& \left.-\left(\frac{H}{S}\right)^{\delta-1}\Phi\left[d_{bs}\left(\frac{H^2}{S_t},\max\{H,K\}\right)\right]\right\},
\end{aligned}
\tag{4.23}
$$

and

$$
DR(t,S) = \left(\frac{H}{S_t}\right)^{\alpha_+}\Phi\left[\theta\frac{\ln\left(\frac{H}{S_t}\right)+\beta\tau}{\sigma\sqrt{\tau}}\right] + \left(\frac{H}{S_t}\right)^{\alpha_-}\Phi\left[\theta\frac{\ln\left(\frac{H}{S_t}\right)-\beta\tau}{\sigma\sqrt{\tau}}\right], \tag{4.24}
$$

where $\Phi[.]$ is the normal cumulative distribution function,

$$
d_{bs}(Y,X) = \frac{\ln(Y/X)+(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}},
$$

$$
d_{bs1}(Y,X) = d_{bs}(Y,X)+\sigma\sqrt{\tau},
$$

$$
\delta = \frac{2(r-q)}{\sigma^2},
$$

$$
\theta = \mathrm{sign}\left[\ln\left(\frac{S_t}{H}\right)\right],
$$

$$
\beta = \sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2+2r\sigma^2},
$$

$$
\alpha_\pm = \frac{r-q-\frac{\sigma^2}{2}\pm\beta}{\sigma^2},
$$

and all contracts are initiated at time zero, $m_{0,t}$ and $M_{0,t}$ are the minimum and maximum prices recorded until date t, S_t is the current underlying asset price at time t and $\tau = T-t$ is the time remaining to expiration.

Chapter 5

CEV Model

Empirical evidence supports the hypothesis that the volatility changes with the stock price. The plot of implied volatility, known as *volatility smile* (see Hull, 2002, figure 15.1) evidences that the volatility of the asset is related to the strike price. This effect of *volatility smile* is not captured by the Black and Scholes (1973) model, since one of the assumptions of this model is that the volatility of the underlying asset price is constant. The constant elasticity of variance (CEV) model of Cox (1975) allows the instantaneous conditional variance of asset returns to depend on the asset price level, thereby displaying an implied *volatility smile* (or skew) similar to the volatility observed in practice. The lognormal assumption with constant volatility does not capture the the so-called *leverage effect* (i.e., the existence of a negative correlation between stock returns and realized stock volatility) observed across a large range of markets and underlying assets. The CEV framework is consistent with leverage effect.

The comparison between the Black and Scholes (1973) model and the CEV model has been made by several authors. MacBeth and Merville (1980) concluded that the prices obtained by the CEV model were closer to market quotes than the ones obtained by the Black and Scholes (1973) model, specially in case of $\beta < 0$. Boyle and Tian (1999) concluded that the difference of prices between the two models is greather for path-dependent options than for standard options.

The CEV model assumes that the risk-neutral process for a stock price, S_t, is

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \delta S_t^{\beta+1} \mathrm{d}W_t^{\mathcal{Q}} \qquad t \geq 0, \qquad S_0 = S > 0, \tag{5.1}$$

where $\mathcal{Q}$ is an equivalent martingale measure (risk-neutral probability measure). Under $\mathcal{Q}$, we suppose that the asset price is a time-homogeneous, nonnegative diffusion process; $\{W_t^Q : t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{Q})$ and μ is a constant ($\mu = r - q$, where $r \geq 0$ and $q \geq 0$ are the constant risk-free interest rate and the constant dividend yield, respectively).

The CEV model acquired its name from the fact that the elasticity of variance of the rate of return on S_t is constant. This means that the ratio of any proportional change in S_t and the resulting proportional change in the variance of the rate of return on S_t will be constant. Taking the ratio of these two quantities, and noting the variance of the rate of return on S_t is $\mathrm{Var}\left(\frac{\mathrm{d}S_t}{S_t}\right) = \delta^2 S_t^{2\beta}\mathrm{d}t$, yields the following relationship:

$$\begin{aligned}
\frac{\frac{\mathrm{dVar}\left(\frac{\mathrm{dS_t}}{\mathrm{S_t}}\right)}{\mathrm{Var}\left(\frac{\mathrm{d}S_t}{S_t}\right)}}{\frac{\mathrm{d}S_t}{S_t}} &= \frac{\mathrm{dVar}\left(\frac{\mathrm{dS_t}}{\mathrm{S_t}}\right)}{\mathrm{d}S_t}\frac{S_t}{\mathrm{Var}\left(\frac{\mathrm{d}S_t}{S_t}\right)} \\
&= (2\beta\delta^2 S_t^{2\beta-1}\mathrm{d}t)\frac{S_t}{\delta^2 S_t^{2\beta}\mathrm{d}t} \\
&= 2\beta\,. \qquad (5.2)
\end{aligned}$$

Hence, the local volatility $\sigma(S_t) = \delta S_t^{\beta}$ increases as the asset price increases when $\beta > 0$. Note that when beta is less than zero then there is a decrease in the volatility when the asset price increases.

The two model parameters β and δ can be interpreted as the elasticity of the local volatility $\left(\frac{\mathrm{d}\sigma(S_t)}{\mathrm{d}S_t} = \beta\frac{\sigma(S_t)}{S_t}\right)$ and the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma(S_0) = \delta S_0^{\beta}$.

Cox (1975) originally studied the case $\beta < 0$ and Emanuel and MacBeth (1982) consider the CEV process when $\beta > 0$. Since the equity markets usually exhibit volatility skews of negative slope, the CEV process with $\beta > 0$ is rarely considered in the literature.

Unlike the geometric Brownian motion, a solution of equation (5.1) can become negative, unless a constraint is imposed. This is clearly inappropriate for an asset price; therefore an absorbing barrier should be imposed at zero, such that if $S_t = 0$ then $S_u = 0$ for all $u > t$.

From equation (5.1) we can get other option pricing models as special cases:

- When $\beta = 0$ (zero elasticity case), the CEV model is the lognormal model of Black and Scholes (1973);
- When $\beta = -\frac{1}{2}$ it yields the Cox and Ross (1976) square-root model;
- When $\beta = -1$ it corresponds to the Cox and Ross (1976) absolute model.

5.1 Lookback options under the CEV Model

To compute the prices of lookback options under the CEV model, we need to obtain the solutions for the functions ϕ_λ and ψ_λ defined in Lemma 1 at Chapter 2 under the CEV diffusion --- see Davydov and Linetsky (2001, Appendix A) or Ferreira (2009, Appendix A) for the proof of the proposition.

Proposition 6. *Suppose $\beta \neq 0$ and $\lambda > 0$ in the diffusion* (5.1). *The fundamental increasing ($\psi_\lambda(S)$) and decreasing ($\phi_\lambda(S)$) solutions of the CEV ordinary differential equation (ODE)*

$$\frac{1}{2}\sigma^2(S)S^2\frac{\mathrm{d}^2u}{\mathrm{d}S^2} + \mu S\frac{\mathrm{d}u}{\mathrm{d}S} - \lambda u = 0 \ , \quad S \in (0,\infty) \tag{5.3}$$

are (up to multiplicative constants):

$$\psi_\lambda(S) = \begin{cases} S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(S)}M_{k,m}(x(S)), & \beta < 0, \mu \neq 0, \\ S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(S)}W_{k,m}(x(S)), & \beta > 0, \mu \neq 0, \\ S^{\frac{1}{2}}\mathcal{I}_\nu(\sqrt{2\lambda}z(S)), & \beta < 0, \mu = 0, \\ S^{\frac{1}{2}}\mathcal{K}_\nu(\sqrt{2\lambda}z(S)), & \beta > 0, \mu = 0, \end{cases} \tag{5.4}$$

and

$$\phi_\lambda(S) = \begin{cases} S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(S)}W_{k,m}(x(S)), & \beta < 0, \mu \neq 0, \\ S^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(S)}M_{k,m}(x(S)), & \beta > 0, \mu \neq 0, \\ S^{\frac{1}{2}}\mathcal{K}_\nu(\sqrt{2\lambda}z(S)), & \beta < 0, \mu = 0, \\ S^{\frac{1}{2}}\mathcal{I}_\nu(\sqrt{2\lambda}z(S)), & \beta > 0, \mu = 0, \end{cases} \tag{5.5}$$

where $M_{k,m}(x(S))$ and $W_{k,m}(x(S))$ are the Whittaker functions defined in Abramowitz and Stegun (1972, p.504), and $\mathcal{I}_\nu(\sqrt{2\lambda}z(S))$ and $\mathcal{K}_\nu(\sqrt{2\lambda}z(S))$ are the modified Bessel functions also defined in Abramowitz and Stegun (1972, p.374), with:

$$x(S) = \frac{|\mu|}{\delta^2\,|\beta|}S^{-2\beta}, \tag{5.6}$$

$$z(S) = \frac{1}{\delta\,|\beta|}S^{-\beta}, \tag{5.7}$$

$$\epsilon = \mathrm{sign}(\mu\beta) = \begin{cases} 1 & \Leftarrow \mu\beta > 0 \\ -1 & \Leftarrow \mu\beta < 0 \end{cases}, \tag{5.8}$$

$$m = \frac{1}{4\,|\beta|}, \tag{5.9}$$

$$k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{\lambda}{2\,|\mu\beta|}, \tag{5.10}$$

$$\nu = \frac{1}{2\,|\beta|}. \tag{5.11}$$

Moreover, the Wronskian of the functions ψ_λ *and* ϕ_λ *with respect to the scale density of the CEV diffusion,*[1] *is*

$$\omega_\lambda = \begin{cases} \frac{2|\mu|\Gamma(2m+1)}{\delta^2\Gamma(m+k-\frac{1}{2})} & \Leftarrow \mu \neq 0, \\ |\beta| & \Leftarrow \mu = 0, \end{cases} \tag{5.12}$$

where $\Gamma(x)$ *is the Euler Gamma function.*

The lookback options are priced by equations (2.10) - (2.11) and (2.18) - (2.21), replacing the functions ϕ_λ and ϕ_λ by the equations (5.4) and (5.5), respectively. The functions $F(y; S_t, \tau)$ and $G(y; S_t, \tau)$ are obtained by inverting the Laplace transform of equations (2.10) and (2.11), respectively, and the integrals in y in equations (2.18) - (2.21) are computed numerically.

[1]See Davydov and Linetsky (2001) for the explicit formula of the scale density of the CEV diffusion

5.2 Turbo Warrants under the CEV Model

The explicit solution of turbo warrants under the CEV model can be obtained by replacing functions ψ_λ and ϕ_λ as given by equations (5.4) and (5.5) into equations (3.31),[2] (3.32) and (3.36). For the *DOC* component we have to find the expression of speed density of the diffusion (5.1) to compute in closed form the integrals (3.24) and (3.25).

The speed density of the diffusion is

$$m(S) = \frac{2}{\delta^2 S^{2\beta+2}} \exp(-\epsilon x(S)), \tag{5.13}$$

where $x(S)$ and ϵ are defined in equations (5.6) and (5.8). The proof can be seen in (Ferreira, 2009, Appendix A).

The closed-form solutions for integrals I_λ and J_λ, which are defined in equations (3.24) and (3.25), respectively, for the CEV model, are given by (see Ferreira (2009, Appendix B) for details):

$$
\begin{aligned}
&I_\lambda(K, A, B) \\
&= \begin{cases}
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{Y^{1/2}}{2m+1}\exp\big(\frac{x(Y)}{2}\big)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big) & \\
\qquad -\frac{2mKY^{-1/2}}{m-k-\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu>0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{Y^{1/2}}{2m+1}\exp\big(-\frac{x(Y)}{2}\big)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big) & \\
\qquad +\frac{2mKY^{-1/2}}{m+k-\frac{1}{2}}\exp\big(-\frac{x(Y)}{2}\big)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu<0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[Y^{1/2}\exp\big(-\frac{x(Y)}{2}\big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big) & \\
\qquad +KY^{-1/2}\exp\big(-\frac{x(Y)}{2}\big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu>0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{Y^{1/2}}{k-m+\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big) & \\
\qquad -\frac{KY^{-1/2}}{m+k+\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu<0 \\
\frac{2}{\delta\sqrt{2\lambda}}\Big[Y^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(Y)\big) & \\
\qquad -KY^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu=0 \\
\frac{2}{\delta\sqrt{2\lambda}}\Big[Y^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(Y)\big) & \\
\qquad -KY^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu=0
\end{cases}
\end{aligned}
\tag{5.14}
$$

[2] See Proposition 7 in page 33 for the proof of conditions of Proposition 4.

$$
J_\lambda(K,A,B) = \begin{cases}
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{Y^{1/2}}{k+m+\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big) \\
\qquad -\frac{KY^{-1/2}}{k-m+\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu>0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[Y^{1/2}\exp\big(-\frac{x(Y)}{2}\big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big) \\
\qquad +KY^{-1/2}\exp\big(-\frac{x(Y)}{2}\big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu<0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{2mY^{1/2}}{k+m-\frac{1}{2}}\exp\big(-\frac{x(Y)}{2}\big)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big) \\
\qquad +\frac{KY^{-1/2}}{2m+1}\exp\big(-\frac{x(Y)}{2}\big)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu>0 \\
\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[\frac{2mY^{1/2}}{k-m+\frac{1}{2}}\exp\big(\frac{x(Y)}{2}\big)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(Y)\big) \\
\qquad +\frac{KY^{-1/2}}{2m+1}\exp\big(\frac{x(Y)}{2}\big)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu<0 \\
\frac{2}{\delta\sqrt{2\lambda}}\Big[-Y^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(Y)\big) \\
\qquad +KY^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(Y)\big)\Big]_A^B & \Leftarrow \beta<0\wedge\mu=0 \\
\frac{2}{\delta\sqrt{2\lambda}}\Big[-Y^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(Y)\big) \\
\qquad +KY^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(Y)\big)\Big]_A^B & \Leftarrow \beta>0\wedge\mu=0
\end{cases} \tag{5.15}
$$

where $M_{k,m}(x(Y))$ and $W_{k,m}(x(Y))$ are the Whittaker functions defined in Abramowitz and Stegun (1972, p.504), and $\mathcal{I}_\nu(\sqrt{2\lambda}z(Y))$ and $\mathcal{K}_\nu(\sqrt{2\lambda}z(Y))$ are the modified Bessel functions also defined in Abramowitz and Stegun (1972, p.374), with $x(Y)$ and $z(Y)$ defined in equations (5.6) and (5.7), respectively.

To apply the equation (3.31) we need to prove that the conditions in equation (3.30) are true. Next proposition give us that proof and the closed-form for $\lim_{U\to\infty} J_\lambda(K,S,U)$.

Proposition 7. *Under CEV process, for any $S>0$ and for any $\lambda>0$ such that*

$$
\begin{cases}
-2\beta\left(-\frac{1}{2}-\frac{1}{4\beta}-\frac{\lambda}{2|\mu\beta|}-\frac{1}{2}\right)+\frac{1}{2}<0 & \Leftarrow \beta<0\wedge\mu>0 \\
\lambda>0 & \Leftarrow \beta>0\vee\mu\leq 0
\end{cases}, \tag{5.16}
$$

we have

$$
\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)=0\,, \tag{5.17}
$$

and for $\lambda > 0$ such that

$$\begin{cases} 1 - \frac{\lambda}{|\mu|} < 0 & \Leftarrow \beta < 0 \wedge \mu > 0 \\ \lambda > 0 & \Leftarrow \beta > 0 \vee \mu \leq 0 \end{cases}, \tag{5.18}$$

we get

$$\lim_{U\to\infty} J_\lambda(K,S,U) =$$

$$= \begin{cases} -\frac{1}{\delta\sqrt{|\beta\mu|}} e^{\frac{x(S)}{2}} \Big[\frac{S^{1/2}}{k+m+\frac{1}{2}} W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\ \qquad - \frac{KS^{-1/2}}{k-m+\frac{1}{2}} W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \Big] & \Leftarrow \mu > 0 \wedge \beta < 0\,, \\ -\frac{1}{\delta\sqrt{|\beta\mu|}} e^{-\frac{x(S)}{2}} \Big[S^{1/2} W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\ \qquad + KS^{-1/2} W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \Big] & \Leftarrow \mu < 0 \wedge \beta < 0\,, \\ \frac{1}{\delta\sqrt{|\beta\mu|}} \Big[\frac{2m}{k+m-\frac{1}{2}} \Big(\frac{|\mu|}{\delta^2|\beta|}\Big)^m - \frac{2mS^{1/2}}{k+m-\frac{1}{2}} e^{-\frac{x(S)}{2}} M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\ \qquad - \frac{KS^{-1/2}}{2m+1} e^{-\frac{x(S)}{2}} M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Big] & \Leftarrow \mu > 0 \wedge \beta > 0\,, \\ \frac{1}{\delta\sqrt{|\beta\mu|}} \Big[\frac{2m}{k-m+\frac{1}{2}} \Big(\frac{|\mu|}{\delta^2|\beta|}\Big)^m - \frac{2mS^{1/2}}{k-m+\frac{1}{2}} e^{\frac{x(S)}{2}} M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\ \qquad - \frac{KS^{-1/2}}{2m+1} e^{\frac{x(S)}{2}} M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Big] & \Leftarrow \mu < 0 \wedge \beta > 0\,, \\ \frac{2}{\delta\sqrt{2\lambda}} \Big[S^{1/2-\beta} \mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S) \\ \qquad - KS^{-1/2-\beta} \mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\big) \Big] & \Leftarrow \mu = 0 \wedge \beta < 0\,, \\ \frac{2}{\delta\sqrt{2\lambda}} \Big[\frac{1}{\Gamma(\nu)} \Big(\frac{\sqrt{2\lambda}}{2\delta|\beta|}\Big)^{\nu-1} + S^{1/2-\beta} \mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big) \\ \qquad - KS^{-1/2-\beta} \mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big) \Big] & \Leftarrow \mu = 0 \wedge \beta > 0\,. \end{cases} \tag{5.19}$$

where $M_{k,m}(x(S))$ and $W_{k,m}(x(S))$ are the Whittaker functions defined in Abramowitz and Stegun (1972, p.504), and $\mathcal{I}_\nu(\sqrt{2\lambda}z(S))$ and $\mathcal{K}_\nu(\sqrt{2\lambda}z(S))$ are the modified Bessel functions also defined in Abramowitz and Stegun (1972, p.374) with $x(Y)$ and $z(Y)$ are defined in equations (5.6) *and* (5.7)*, respectively.*

Proof. See Appendix E. ☐

Therefore, to compute the *DOC* component of the turbo call we need to replace ψ_λ, ϕ_λ, ω_λ, I_λ, J_λ and $\lim_{x\to\infty} J_\lambda(A,B,x)$, defined (for this model) in equations (5.4), (5.5), (5.12), (5.14), (5.15) and (5.19), into equation (3.31), using the definition of $\Delta(A, B)$ given in equation (3.23). After that, we have to invert the Laplace transform of (3.31) and multiply the result by $e^{-r(T-t)}$.

For the LB component we have to invert the Laplace transform of equation (2.10), replacing ϕ_λ defined in equation (5.5) into equation (2.10) to compute $F(y; H, T_0)$. The result must be replace in equation (3.32).

The DR value is obtained by inverting the Laplace transform of equation (3.36), replacing ϕ_λ defined in equation (5.5) into equation (3.36).

Then we have to replace the values of DOC, LB and DR into equation (3.15).

Chapter 6

Numerical Analysis

Before presenting the numerical results for turbo warrants, it is necessary to explain the method used to invert the Laplace transform of equations (2.10), (3.31) and (3.36) which is required for pricing turbo warrants.

6.1 Euler algorithm

We follow Abate and Whitt (1995, p.37-39) to invert the Laplace transform of equations (2.10), (3.31) and (3.36). The method described is called the *Euler method* by the authors because this method employs Euler summation. Euler summation is one of the more elementary acceleration techniques. The *Euler method* is based on the Bromwich contour inversion integral, which can be expressed as the integral of a real-valued function of a real variable by choosing a specific contour.

The objective of the *Euler method* is to calculate values of a real-valued function $f(t)$ of a positive variable t for various values of t from the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t \,,$$

where s is a complex variable with a nonnegative real part. Its inversion formula is defined as follows:

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s) \mathrm{d}s \,,$$

where $a > 0$ is arbitrary, but must be chosen so that it is greater than the real parts of all the singularities of $F(s)$.

Following Abate and Whitt (1995), an approximation of $f(t)$ can be obtained by

$$f(t) \approx \frac{e^{A/2}}{2t}\mathrm{Re}(F)\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t}\sum_{k=1}^{\infty}(-1)^k \mathrm{Re}(F)\left(\frac{A+2k\pi i}{2t}\right), \tag{6.1}$$

where $\mathrm{Re}(s)$ is the real part of s and $A = 2ta$. The approximation above involves a discretization error. As $|f(t)| \leq 1$ for all t, then this error is bonded by

$$|e_d| \leq \frac{e^{-A}}{1-e^{-A}},$$

which is approximately equal to e^{-A} when this quantity is small. If we need a discretization error less than $10^{-\gamma}$, then we let $A = \gamma \log(10)$. In this thesis we use $A = 8\log(10)$ to achieve a 10^{-8} discretization error.

To apply the *Euler method*, we need to truncate the infinite series in equation (6.1) to n terms. Let $s_n(t)$ be this truncation, i.e:

$$s_n(t) = \frac{e^{A/2}}{2t}\mathrm{Re}(F)\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t}\sum_{k=1}^{n}(-1)^k \mathrm{Re}(F)\left(\frac{A+2k\pi i}{2t}\right). \tag{6.2}$$

Then we apply the Euler summation to m terms after an initial n, i.e. we compute the binomial average of the terms s_n, s_{n+1}, ..., s_{n+m}:

$$E(m,n,t) = \sum_{k=0}^{m}\binom{m}{k}2^{-m}s_{n+k}(t). \tag{6.3}$$

So, equation (6.3) is an approximation to equation (6.1).

6.2 Numerical Results

6.2.1 Lookback Options

Using the pricing formulae derived in sections 4.1 and 5.1 and the Euler algorithm explained in section 6.1, we obtain the prices of lookback options under the GBM and CEV models, that are illustrated in Tables 6.1 and 6.2. The first table contains the prices computed with Matlab(R2010) and the second table contains the prices obtained with Mathematica(8.0). In all tables, the number of extraterms used in Euler algorithm is $m = 11$.

We adopted the same choice of parameters as Davydov and Linetsky (2001): the initial asset price is $S_0 = S = 100$; the instantaneous volatility at this price level is $\sigma_0 = \sigma = 25\%$ per annum; the risk-free interest rate is 10% per annum; the asset price pays no dividend; and all options have six months to expiration ($T = 0.5$). In the CEV model, we deploy seven values of β to show its effect on lookback options prices: $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. Following Davydov and Linetsky (2001), and to ensure that option prices based on different values of β are comparable, the value of δ is readjusted so that the initial instantaneous volatility is the same across different models. Then, the value of δ to be used for the CEV model with different β values is adjusted to $\delta = \sigma_0 S_0^{-\beta}$.

Examination of Tables 6.1 and 6.2 reveals that floating strike lookback call and fixed strike lookback put options under the CEV model with negative (positive) β are worth more (less) than under the GBM model. In contrast, floating strike lookback put and fixed strike lookback call options are worth less (more) under the CEV model with negative (positive) β than under the GBM model. However, the prices of these contracts when β is equal to 0.5 are nonsensical. Analysing the two tables we can conclude that the larger the absolute value of β, the greater the price difference between CEV and GBM option prices. In the CEV model, the value of β has a greater impact on the prices of lookback options. Thus, a misspecified value of β may cause very large pricing errors.

Comparing the prices of lookback options under the GBM model, we notice that values obtained using the Laplace transform are equal to values obtained with Zhang (1998) formulae. However, if we use Zhang (1998) formulae, the computation time decreases.

If we compare the results of both softwares (Tables 6.1 and 6.2), the prices of lookback options are almost equals, but we can observe some differences. The prices of fixed strike lookback call and floating strike lookback put for the CEV model when $\beta = -4$ could not be obtained with Matlab(R2010). The prices of floating strike lookback call and fixed strike lookback put when $\beta = -0.5$ could not be obtained with Matlab(R2010) or Mathematuca(8.0). Also in the CEV model, when $\beta = -1$, the prices obtained with Matlab differ from the prices obtained with Mathematica by more than one tenth. However, comparing our results with Davydov and Linetsky (2001, Table 2), their prices of lookback options are closer to our prices obtained with Mathematica (the difference is less than one hundredth, in this case).

Table 6.1: Prices of lookback options under the CEV and GBM models, computed via Matlab(R2010)

		CEV					GBM	
				β				
S	**K**	**-4**	**-3**	**-2**	**-1**	**-0.5**	**LT**	**Zhang**
				Floating strike lookback call				
100	N/A	19.5629 (n = 7)	18.2922 (n = 7)	17.0048 (n = 4)	16.3131 (n = 5)	NaN (n = 7)	15.6357 (n = 15)	15.6357
				Fixed strike lookback call				
100	100	NaN (n = 7)	14.7988 (n = 7)	15.3807 (n = 7)	16.1395 (n = 7)	16.6083 (n = 7)	17.1598 (n = 15)	17.1598
100	105	NaN (n = 7)	10.3599 (n = 7)	10.9823 (n = 7)	11.7748 (n = 7)	12.2588 (n = 7)	12.8246 (n = 15)	12.8246
				Floating strike lookback put				
100	N/A	NaN (n = 7)	9.9217 (n = 7)	10.5037 (n = 7)	11.2624 (n = 7)	11.7312 (n = 7)	12.2828 (n = 15)	12.2828
				Fixed strike lookback put				
100	95	10.7897 (n = 7)	9.4733 (n = 7)	8.1378 (n = 4)	7.0764 (n = 7)	NaN (n = 7)	6.6631 (n = 15)	6.6631
100	100	14.6859 (n = 7)	13.4151 (n = 7)	12.1277 (n = 4)	11.4360 (n = 5)	NaN (n = 7)	10.7587 (n = 15)	10.7587
CPU time (in seconds)				207,994.05			1.6711	0.0017

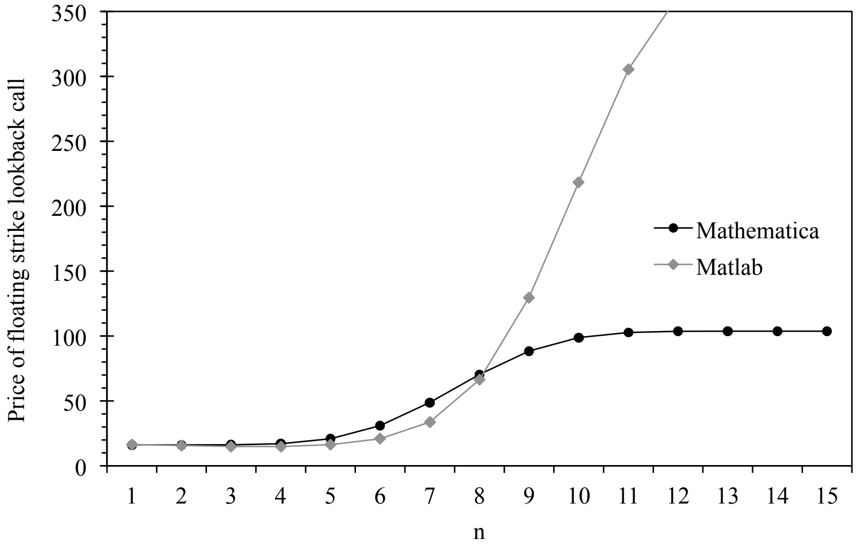
The asset price S and the strike price K vary as indicated in first and second columns, respectively. The prices of lookback options under the CEV model were calculated with different elasticities, $\beta \in \{-4, -3, -2, -1, -0.5\}$. The number of terms n used in the *Euler method* is given in parentheses below the corresponding option price. The penultimate and the last columns contain the prices obtained, under the GBM model, using the Laplace Transform (LT) and Zhang (1998) formulae (with the changes made when $r = q$), respectively. The last row indicates the computation time for all contracts using the CEV model, the Laplace transform for the GBM model and using Zhang (1998) formulae. The parameters used in the calculations are: $\sigma = 0.25$, $r = 0.1$, $q = 0$, $T = 0.5$, $m_{0,T} = 100$ and $M_{0,T} = 100$.

Table 6.2: Prices of lookback options under the CEV and GBM models, computed via Mathematica(8.0)

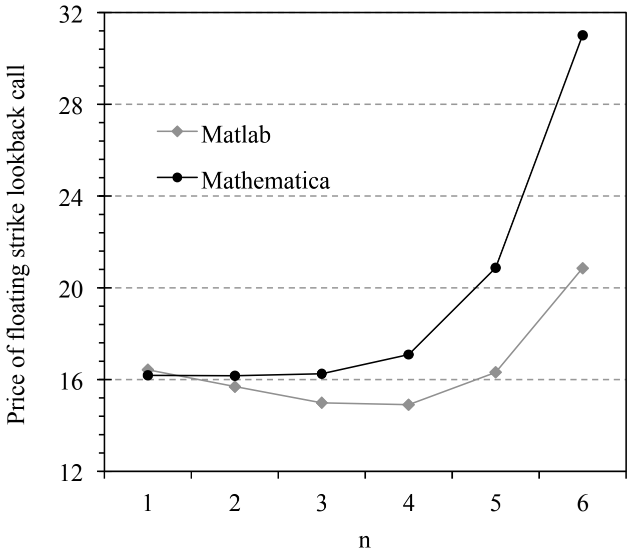
		CEV							GBM	
		β								
S	**K**	**-4**	**-3**	**-2**	**-1**	**-0.5**	**0.5**	**1**	**LT**	**Zhang**
		Floating strike lookback call								
100	N/A	19.5629 (n = 7)	18.2922 (n = 7)	17.0048 (n = 7)	16.1663 (n = 2)	NaN (n = 7)	15.4274 (n = 7)	15.2468 (n = 7)	15.6357 (n = 15)	15.6357
		Fixed strike lookback call								
100	100	14.3562 (n = 7)	14.7988 (n = 7)	15.3807 (n = 7)	16.1395 (n = 7)	16.6084 (n = 7)	0.0000 (n = 7)	18.6657 (n = 1)	17.1598 (n = 15)	17.1598
100	105	9.8669 (n = 7)	10.3599 (n = 7)	10.9823 (n = 7)	11.7748 (n = 7)	12.2588 (n = 7)	0.0000 (n = 7)	14.3260 (n = 3)	12.8246 (n = 15)	12.8246
		Floating strike lookback put								
100	N/A	9.4791 (n = 7)	9.9217 (n = 7)	10.5037 (n = 7)	11.2624 (n = 7)	11.7313 (n = 7)	-4.8771 (n = 7)	13.7886 (n = 1)	12.2828 (n = 15)	12.2828
		Fixed strike lookback put								
100	95	10.7897 (n = 7)	9.4733 (n = 7)	8.1378 (n = 7)	7.2496 (n = 2)	NaN (n = 7)	6.4258 (n = 7)	6.2150 (n = 7)	6.6631 (n = 15)	6.6631
100	100	14.6859 (n = 7)	13.4151 (n = 7)	12.1277 (n = 7)	11.2892 (n = 2)	NaN (n = 7)	10.5503 (n = 7)	10.3698 (n = 7)	10.7587 (n = 15)	10.7587
CPU time (in seconds)		595.33					3,480.73		1.2951	0.2484

The asset price S and the strike price K vary as indicated in first and second columns, respectively. The prices of lookback options under the CEV model were calculated with different elasticities, $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. The number of terms n used in the *Euler method* is given in parentheses below the corresponding option price. The penultimate and the last columns contain the prices obtained, under the GBM model, using the Laplace Transform (LT) and Zhang (1998) formulae (with the changes made when $r = q$), respectively. The last row indicates the computation time for all contracts using the CEV model, the Laplace transform for the GBM model and using Zhang (1998) formulae. The parameters used in the calculations are: $\sigma = 0.25$, $r = 0.1$, $q = 0$, $T = 0.5$, $m_{0,T} = 100$ and $M_{0,T} = 100$.

Figure 6.1: Price of a floating strike lookback call under the CEV model when $\beta = -1$, and for different values of n (number of terms used in the *Euler method*), using Matlab(R2010) and Mathematica(8.0).



(a)



(b)

The parameters used in the calculations are: $S = 100$, $\sigma = 0.25$, $r = 0.1$, $q = 0$, $T = 0.5$, $m_{0,T} = 100$ and $\beta = -1$. Graph (b) is a zoom of graph (a).

Figure 6.1 illustrates the relationship between the price of a floating strike lookback call, under the CEV model with $\beta = -1$, and the number of terms in the Euler algorithm (n). Observing Graph 6.2(a), we can notice that if n is a large number, the price of a floating strike lookback call is nonsensical. In Figure 6.2(b), we observe that a misspecified value of n may cause significant pricing errors, using Matlab(R2010) or Mathematica(8.0). The same occurs also for fixed strike lookback put options when $\beta = -1$. For the case $\beta = -2$, the specification of n is important for floating strike lookback call or fixed strike lookback put options, when we use Matlab(R2010). The number of terms in the Euler algorithm (n) used to compute the prices of fixed strike lookback call and floating strike lookback put options shown in Table 6.2 when $\beta = 1$ were chosen so that the values make sense. So, the methodology proposed by Davydov and Linetsky (2001) is not the best tool to compute the price of lookback options due to the fact that if we want to know the price of these contracts, we have to compute them with a large enough value of n for prices to converge, and we proved that, in some cases, the price of lookback options converges to a nonsensical value, so we have to choose a small value for n.

6.2.2 Turbo Warrants

Using the pricing formulae derived in sections 4.2 and 5.2 and the Euler algorithm explained in section 6.1, we obtain the prices of turbo call warrants under the GBM and CEV models, that are illustrated in Tables 6.3 - 6.8. In all tables, the number of extraterms used in the Euler algorithm is $m = 11$.

Table 6.3: Prices of turbo call warrants computed under the GBM model and using Matlab(R2010) when $r = q = 5\%$

			$\sigma = 20\%$		$\sigma = 60\%$	
K	**H**	$\mathbf{T_0}$	**LT**	**Zhang**	**LT**	**Zhang**
8	9	0.2	1.5739	1.5739	1.1753	1.1753
8	9	0.8	1.4402	1.4402	1.0980	1.0980
7	9	0.2	2.4998	2.4998	1.6735	1.6735
7	9	0.8	2.1823	2.1823	1.3666	1.3666
7	8	0.2	2.7106	2.7106	2.2350	2.2350
7	8	0.8	2.6443	2.6443	2.1596	2.1596
6	8	0.2	3.6541	3.6541	2.8328	2.8328
6	8	0.8	3.5125	3.5125	2.5379	2.5379
CPU time (in seconds)			0.0500	0.0049	0.0540	0.0058

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The fourth and sixth columns contain the prices obtained under the GBM model, using the Laplace Transform (LT) for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The fifth and seventh columns contain the prices obtained under the GBM model, using Zhang (1998) formulae (with the changes made when $r = q$), for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The last row indicates the computation time for all contracts using the Laplace transform and Zhang (1998) formulae. The parameters used in the calculations are: $S = 10$, $r = 5\%$, $q = 5\%$ and $T = 1$. The number of terms n used in the Euler method is 7.

For the GBM model, the parameters adopted are: the initial asset price is $S_0 = S = 10$; the instantaneous volatility at this price level is $\sigma = 20\%$ per annum or $\sigma = 60\%$; and all contracts have one year to expiration ($T = 1$). In Table 6.3 the risk-free interest rate (r) and the dividend yield (q) are 5%, in Table 6.4 these parameters are $r = 5\%$ and $q = 0\%$, and in Table 6.5 we have $r = 0\%$ and $q = 5\%$.

Examination of the results for the GBM model (Tables 6.3-6.5) reveals that when increasing the volatility (σ), the price of the turbo call warrant is reduced. We also vary the strike price ($K \in \{6, 7, 8\}$) and the barrier level ($H = 8$ or $H = 9$). When we decrease the strike price (K) or the barrier level (H), the price of turbo call warrant increases.

Table 6.4: Prices of turbo call warrants computed under the GBM model and using Matlab(R2010) when $r = 5\%$ and $q = 0\%$

			$\sigma = 20\%$		$\sigma = 60\%$	
K	**H**	$\mathbf{T_0}$	**LT**	**Zhang**	**LT**	**Zhang**
8	9	0.2	1.9582	1.9582	1.3366	1.3366
8	9	0.8	1.8508	1.8508	1.2602	1.2602
7	9	0.2	2.8933	2.8933	1.8509	1.8509
7	9	0.8	2.6539	2.6539	1.5511	1.5511
7	8	0.2	3.2019	3.2019	2.5204	2.5204
7	8	0.8	3.1565	3.1565	2.4470	2.4470
6	8	0.2	4.1490	4.1490	3.1365	3.1365
6	8	0.8	4.0578	4.0578	2.8531	2.8531
CPU time (in seconds)			0.0411	0.0050	0.0415	0.0052

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The fourth and sixth columns contain the prices obtained under the GBM model, using the Laplace Transform (LT) for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The fifth and seventh columns contain the prices obtained under the GBM model, using Zhang (1998) formulae, for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The last row indicates the computation time for all contracts using the Laplace transform and Zhang (1998) formulae. The parameters used in the calculations are: $S = 10$, $r = 5\%$, $q = 0\%$ and $T = 1$. The number of terms n used in the Euler method is 7.

These results make sense due to the fact that if the barrier is never touched, the value of the turbo call warrant is greater. So, if we increase the volatility (or the strike price or the barrier level), then the probability of the asset price (S) touching the barrier level is greater. Therefore, the price of the turbo call warrant decreases. For the parameter T_0, we choose two values to compare the results: $T_0 = 0.2$ and $T_0 = 0.8$. This parameter is the time to maturity of the LB component of the turbo call warrant (that starts when the barrier level is touched), and is also an important factor because if we increase T_0, the price of a turbo call warrant decreases. This is explained by the increasing of the probability of the minimum of asset price be less than the strike price (K), because the time to expiration is bigger.

In the GBM model, the prices of turbo call warrants obtained using the Laplace transform and Zhang (1998) formulae are equal, but the second method is the fastest. Comparing the three tables with the GBM model's results (Tables 6.3 - 6.5), we notice that increasing the *drift* ($\mu = r - q$), increases the price of turbo call warrants.

Table 6.5: Prices of turbo call warrants computed under the GBM model and using Matlab(R2010) when $r = 0\%$ and $q = 5\%$

			$\sigma = 20\%$		$\sigma = 60\%$	
K	**H**	$\mathbf{T_0}$	**LT**	**Zhang**	**LT**	**Zhang**
8	9	0.2	1.3057	1.3057	1.0798	1.0798
8	9	0.8	1.1460	1.1460	1.0021	1.0021
7	9	0.2	2.2512	2.2512	1.5728	1.5728
7	9	0.8	1.8513	1.8513	1.2612	1.2612
7	8	0.2	2.3722	2.3722	2.0743	2.0743
7	8	0.8	2.2797	2.2797	1.9969	1.9969
6	8	0.2	3.3542	3.3542	2.6722	2.6722
6	8	0.8	3.1468	3.1468	2.3663	2.3663
CPU time (in seconds)			0.0419	0.0066	0.0503	0.0088

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The fourth and sixth columns contain the prices obtained under the GBM model, using the Laplace Transform (LT) for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The fifth and seventh columns contain the prices obtained under the GBM model, using Zhang (1998) formulae, for $\sigma = 0.2$ and $\sigma = 0.6$, respectively. The last row indicates the computation time for all contracts using the Laplace transform and Zhang (1998) formulae. The parameters used in the calculations are: $S = 10$, $r = 0\%$, $q = 5\%$ and $T = 1$. The number of terms n used in the Euler method is 7.

For the CEV model, the parameters adopted are: the initial asset price is $S_0 = S = 10$; the instantaneous volatility at this price level is $\sigma = 30\%$; the time to maturity of the LB component is $T_0 = 0.8$; and all contracts have one year to expiration ($T = 1$). We also vary the risk-free interest rate (r) and the dividend-yield (q), studying three cases: $r = 5\%$ and $q = 5\%$, $r = 5\%$ and $q = 0\%$, or $r = 0\%$ and $q = 5\%$. We deploy seven values of β to show its effect on lookback options prices: $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. As for lookback options, the value of δ is readjusted so that the initial instantaneous volatility is the same across different models. Then, the value of δ to be used for the CEV model with different β values is adjusted to $\delta = \sigma S_0^{-\beta}$.

Examination of Tables 6.6 and 6.7 reveals that, in the CEV model with $\mu \geq 0$ ($\mu = r - q$), when decreasing the strike price (K) or the barrier level (H), the price is increased. This is explained by the decreasing of the probability of the asset price (S) touch the barrier (H). There is no *a priori* rule to the impact on a turbo call price when β increases or decreases. The value of β has a small impact on the prices of the turbo warrants; however, a misspecified value of this parameter may cause pricing errors,

Table 6.6: Prices of turbo call warrants computed under the CEV model and using Matlab(R2010) when $r = q = 5\%$

			β						
K	**H**	**T_0**	**-4**	**-3**	**-2**	**-1**	**-0.5**	**0.5**	**1**
8	9	0.8	1.3531	1.3374	1.3206	1.3028	1.2934	1.2737	1.2710
7	9	0.8	1.9057	1.8831	1.8627	1.8467	1.8422	1.8408	1.8529
7	8	0.8	2.4567	2.4492	2.4440	2.4432	2.4453	2.4568	2.4754
6	8	0.8	3.1005	3.0928	3.0949	3.1147	3.1344	3.1989	3.2518
CPU time (in seconds)						2.7223			

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The prices of turbo call warrants under the CEV model were calculated with different elasticities, $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. The last row indicates the computation time for all contracts using the CEV model. The number of terms n used in the Euler method is 7. The parameters used in the calculation are: $S = 10, \sigma = 0.30$, $T = 1$, $r = 5\%$ and $q = 5\%$.

Table 6.7: Prices of turbo call warrants computed under the CEV model and using Matlab(R2010) when $r = 5\%$ and $q = 0\%$

			β						
K	**H**	**T_0**	**-4**	**-3**	**-2**	**-1**	**-0.5**	**0.5**	**1**
8	9	0.8	1.6729	1.6519	1.6298	1.6065	NaN	NaN	3.9304
7	9	0.8	2.2756	2.2481	2.2231	2.2035	NaN	NaN	4.2727
7	8	0.8	2.8808	2.8752	2.8736	2.8789	NaN	NaN	-2,245.1*
6	8	0.8	3.5680	3.5628	3.5696	3.5969	NaN	NaN	-2,238.25*
CPU time (in seconds)						3,160.84			

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The prices of turbo call warrants under the CEV model were calculated with different elasticities, $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. The last row indicates the computation time for all contracts using the CEV model. The number of terms n used in the Euler method is 7. The parameters used in the calculation are: $S = 10, \sigma = 0.30$, $T = 1$, $r = 5\%$ and $q = 0\%$.
* These values differ from the ones obtained via Mathematica(8.0), which do not make sense either.

specially in the case when the *drift* is zero ($r = q$), $K = 6$ and $H = 8$. This effect is not as severe as for lookback options, but, it should still be avoided.

In Table 6.7 the price values for $\beta = 1$ are nonsensical. When the absolute value of β

is close to zero, Matlab(R2010) and Mathematica(8.0) programs can not yield any real values (the results with Mathematica are equal to the results from Maltab, so we decided publish only the Matlab results). The results in Table 6.8 are completely absurd.

Table 6.8: Prices of turbo call warrants computed under the CEV model and using Matlab(R2010) when $r = 0\%$ and $q = 5\%$

			β						
K	**H**	$\mathbf{T_0}$	**-4**	**-3**	**-2**	**-1**	**-0.5**	**0.5**	**1**
8	9	0.8	-2.7212	-2.3066	-2.0766	-1.8289	NaN	NaN	0.6350
7	9	0.8	-2.1959	-1.7883	-1.5624	-1.3135	NaN	NaN	0.8951
7	8	0.8	-4.9142	-4.1646	-3.7402	-3.4086	NaN	NaN	-29.8330*
6	8	0.8	-4.2859	-3.5368	-3.1056	-2.7540	NaN	NaN	-30.2768*
CPU time (in seconds)						3,143.47			

The strike price K and the barrier level H vary as indicated in first and second columns, respectively. The third column contains the time to maturity T_0 of the LB component of the turbo call warrant. The prices of turbo call warrants under the CEV model were calculated with different elasticities, $\beta \in \{-4, -3, -2, -1, -0.5, 0.5, 1\}$. The last row indicates the computation time for all contracts using the CEV model. The number of terms n used in the Euler method is 7. The parameters used in the calculation are: $S = 10, \sigma = 0.30$, $T = 1$, $r = 0\%$ and $q = 5\%$.
* These values differ from the ones obtained via Mathematica(8.0).

Chapter 7

Conclusion

This thesis studies the pricing of turbo warrants and lookback options under the GBM and CEV models. Following Davydov and Linetsky (2001), we derived pricing solutions for lookback options in closed-form, and following Wong and Chan (2008), we derived pricing solutions for turbo call warrants also in closed-form.

For the GBM model, we compared two different methods to compute the prices of lookback options: the Laplace transform of the probability distributions of the minimum and maximum asset prices, and the Zhang (1998) formulae.[1] We conclude that both methods produce the same prices, but Laplace transform is the slowest method. We did the same comparison for turbo call warrants: we compared Laplace transforms of the probability distributions of the minimum and maximum asset prices and of the expected value of the terminal payoff of a double knock out option against Zhang (1998) formulae. The conclusion is the same: the Laplace transform methodology is the slowest one.

For the CEV model, the variation of β influences the price of lookback options: when β is increased, the price of floating strike lookback call (put) and fixed strike lookback put (call) options increases (decreases). For turbo warrants, the results did not provide any pattern. Therefore, lookback options and turbo call warrants are sensitive to the specification of the elasticity parameter β, and because of that errors can exist in the price of these contracts if the estimation of β is incorrect.

Although we have obtained closed-form solutions to price these contracts under the CEV model, the results are not good enough: in same cases, we had to choose a number of terms (n) to invert the Laplace transform numerically different from the pre-specified one, because the option price converges to a nonsensical number; in other cases, we did not obtain prices for an absolute value of β near zero; for $\beta > 0$, not all the values made sense; and when the risk-free rate is smaller than the dividend yield ($r < q$), the prices of

[1]However, for $r = q$ this formulae is incorrect, so we had to derive the prices of these contracts.

turbo warrants were nonsensical.

Appendix A

Proof of Proposition 1

Before deriving equations (2.18) to (2.21), we need to follow some steps:

1. We need to show that

$$(X - m_{0,T})^+ = \int_0^X \mathbb{1}_{\{m_{0,T} \leq y\}} \mathrm{d}y \qquad \forall X \in \mathbb{R}^+ . \tag{A.1}$$

Let $y \in [0, X]$ and

$$\mathbb{1}_{\{m_{0,T} \leq y\}} = \begin{cases} 1 & \Leftarrow y \geq m_{0,T} \\ 0 & \Leftarrow y < m_{0,T} \end{cases} .$$

If $X \geq m_{0,T}$, then

$$\begin{aligned} \int_0^X \mathbb{1}_{\{m_{0,T} \leq y\}} \mathrm{d}y &= \int_{m_{0,T}}^X 1 \mathrm{d}y \\ &= X - m_{0,T} . \end{aligned}$$

If $X < m_{0,T}$, then $y \leq X < m_{0,T}$ and $\mathbb{1}_{\{m_{0,T} \leq y\}} = 0$. Thus,

$$\int_0^X \mathbb{1}_{\{m_{0,T} \leq y\}} \mathrm{d}y = 0 .$$

Hence,

$$\begin{aligned} \int_0^X \mathbb{1}_{\{m_{0,T} \leq y\}} \mathrm{d}y &= \begin{cases} X - m_{0,T} & \Leftarrow X \geq m_{0,T} \\ 0 & \Leftarrow X < m_{0,T} \end{cases} \\ &= (X - m_{0,T})^+ . \end{aligned}$$

2. We need to prove that

$$\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] = \int_0^X \mathcal{Q}(m_{0,T} \le y|\mathcal{F}_t)\mathrm{d}y\,. \tag{A.2}$$

Using equation (A.1),

$$\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] = \mathbb{E}_{\mathcal{Q}}\left[\int_0^X \mathbb{1}_{\{m_{0,T}\le y\}}\mathrm{d}y|\mathcal{F}_t\right].$$

Using Fubini's Theorem, the last equation becomes

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] &= \int_0^X \mathbb{E}_{\mathcal{Q}}(\mathbb{1}_{\{m_{0,T}\le y\}}|\mathcal{F}_t)\mathrm{d}y \\ &= \int_0^X \mathcal{Q}(m_{0,T} \le y|\mathcal{F}_t)\mathrm{d}y\,.\end{aligned}$$

3. We need to show that

$$\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] = \begin{cases} \int_0^X F(y;S_t,\tau)\mathrm{d}y & \Leftarrow m_{0,t} \ge X \\ X - m_{0,t} + \int_0^{m_{0,t}} F(y;S_t,\tau)\mathrm{d}y & \Leftarrow m_{0,t} < X \end{cases} \tag{A.3}$$

The minimum price recorded until date T is given by:

$$m_{0,T} = \min\{m_{0,t}, m_{t,T}\}\,.$$

3.1. If $m_{0,t} \ge X$:

Let $y \in [0,X]$. As $m_{0,t} \ge X$ then $m_{0,t} \ge y$. So,

$$\mathcal{Q}(m_{0,T} \le y) = \mathcal{Q}(m_{t,T} \le y)\,. \tag{A.4}$$

Combining equations (A.2) and (A.4) we get

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] &= \int_0^X \mathcal{Q}(m_{t,T} \le y|\mathcal{F}_t)\mathrm{d}y \\ &= \int_0^X F(y;S_t,\tau)\mathrm{d}y\,. \end{aligned} \tag{A.5}$$

3.2. If $m_{0,t} < X$:

Using equation (A.2), then

$$
\begin{aligned}
\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] &= \int_0^X \mathcal{Q}(m_{0,T}\le y|\mathcal{F}_t)\mathrm{d}y \\
&= \int_0^{m_{0,t}} \mathcal{Q}(m_{0,T}\le y|\mathcal{F}_t)\mathrm{d}y + \int_{m_{0,t}}^X \mathcal{Q}(m_{0,T}\le y|\mathcal{F}_t)\mathrm{d}y \\
&= \int_0^{m_{0,t}} \mathcal{Q}(m_{0,T}\le y|\mathcal{F}_t)\mathrm{d}y + \int_{m_{0,t}}^X 1\ \mathrm{d}y\,.
\end{aligned}
$$

The last equality arises because $m_{0,T}\le m_{0,t}$.

Let $y\in[0,m_{0,t}]$. As $m_{0,T}=\min\{m_{0,t},m_{t,T}\}$, then $m_{0,T}\le y \Leftrightarrow m_{t,T}\le y$. Therefore,

$$
\begin{aligned}
\mathbb{E}_{\mathcal{Q}}\left[(X-m_{0,T})^{+}|\mathcal{F}_t\right] &= \int_0^{m_{0,t}} \mathcal{Q}(m_{t,T}\le y|\mathcal{F}_t)\mathrm{d}y + X - m_{0,t} \\
&= \int_0^{m_{0,t}} F(y;S_t,\tau)\mathrm{d}y + X - m_{0,t}\,. \qquad \text{(A.6)}
\end{aligned}
$$

Combining the equations (A.5) and (A.6), equation (A.3) arises.

4. Proof of equations (2.18) and (2.21).
The terminal payoff of a floating strike lookback call on asset S and with expiry date at time T is $LC_{fl}(T;S_T,m_{0,T},T)=S_T-m_{0,T}$. The value at any time $t<T$ of the floating strike lookback call is:

$$
\begin{aligned}
LC_{fl}(t;S_t,m_{0,t},T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[S_T-m_{0,T}|\mathcal{F}_t\right] \\
&= e^{-r\tau}\Big(\mathbb{E}_{\mathcal{Q}}\left[S_T-m_{0,t}|\mathcal{F}_t\right]+\mathbb{E}_{\mathcal{Q}}\left[m_{0,t}-m_{0,T}|\mathcal{F}_t\right]\Big) \\
&= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}(S_T|\mathcal{F}_t) - e^{-r\tau}m_{0,t} + e^{-r\tau}\mathbb{E}_{\mathcal{Q}}(m_{0,t}-m_{0,T}|\mathcal{F}_t) \\
&= e^{-r\tau}S_te^{(r-q)\tau} - e^{-r\tau}m_{0,t} + e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\big[(m_{0,t}-m_{0,T})^{+}|\mathcal{F}_t\big]. \qquad \text{(A.7)}
\end{aligned}
$$

The last equation was obtained due to the fact that $\mathbb{E}_{\mathcal{Q}}(S_T|\mathcal{F}_t)=S_te^{(r-q)\tau}$ and $m_{0,T}\le m_{0,t}$.

Combining equations (A.7) and (A.3) and replacing X for $m_{0,t}$,

$$
LC_{fl}(t;S_t,m_{0,t},T) = e^{-q\tau}S_t - e^{-r\tau}m_{0,t} + e^{-r\tau}\int_0^{m_{0,t}} F(y;S_t,\tau)\mathrm{d}y\,,
$$

and equation (2.18) follows.

The terminal payoff of a fixed strike lookback put on asset S, with strike price K and expiry date at time T is $LP_{fx}(T; S_T, K, m_{0,T}, T) = (K - m_{0,T})^+$.

The value at any time $t < T$ of the fixed strike lookback put is obtained by replacing X for K in equation (A.3) and discounting, which yields equation (2.21).

5. We need to show that

$$(M_{0,T} - X)^+ = \int_X^\infty \mathbb{1}_{\{M_{0,T} \geq y\}} \mathrm{d}y \qquad \forall X \in \mathbb{R}^+ . \tag{A.8}$$

Let $y \in [X, \infty[$ and

$$\mathbb{1}_{\{M_{0,T} \geq y\}} = \begin{cases} 1 & \Leftarrow y \leq M_{0,T} \\ 0 & \Leftarrow y > M_{0,T} \end{cases}$$

If $X \leq M_{0,T}$, then

$$\begin{aligned} \int_X^\infty \mathbb{1}_{\{M_{0,T} \geq y\}} \mathrm{d}y &= \int_X^{M_{0,T}} 1 \mathrm{d}y \\ &= M_{0,T} - X . \end{aligned}$$

If $X > M_{0,T}$, then $y \geq X > M_{0,T}$ and $\mathbb{1}_{\{M_{0,T} \geq y\}} = 0$. Thus,

$$\int_X^\infty \mathbb{1}_{\{M_{0,T} \geq y\}} \mathrm{d}y = 0 .$$

Hence,

$$\begin{aligned} \int_X^\infty \mathbb{1}_{\{M_{0,T} \geq y\}} \mathrm{d}y &= \begin{cases} M_{0,T} - X & \Leftarrow X \leq M_{0,T} \\ 0 & \Leftarrow X > M_{0,T} \end{cases} \\ &= (M_{0,T} - X)^+ . \end{aligned}$$

6. We need to prove that

$$\mathbb{E}_{\mathcal{Q}} \left[(M_{0,T} - X)^+ | \mathcal{F}_t \right] = \int_X^\infty \mathcal{Q}(M_{0,T} \geq y | \mathcal{F}_t) \mathrm{d}y . \tag{A.9}$$

Using equation (A.8),

$$\mathbb{E}_{\mathcal{Q}} \left[(M_{0,T} - X)^+ | \mathcal{F}_t \right] = \mathbb{E}_{\mathcal{Q}} \left[\int_X^\infty \mathbb{1}_{\{M_{0,T} \geq y\}} \mathrm{d}y | \mathcal{F}_t \right] .$$

Using Fubini's Theorem, the last equation becomes

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T}-X)^+|\mathcal{F}_t\right] &= \int_X^\infty \mathbb{E}_{\mathcal{Q}}(\mathbb{1}_{\{M_{0,T}\geq y\}}|\mathcal{F}_t)\mathrm{d}y \\ &= \int_X^\infty \mathcal{Q}(M_{0,T}\geq y|\mathcal{F}_t)\mathrm{d}y\,.\end{aligned}$$

7. We need to show that

$$\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T}-X)^+|\mathcal{F}_t\right] = \begin{cases}\int_X^\infty G(y;S_t,\tau)\mathrm{d}y & \Leftarrow M_{0,t}\leq X \\ M_{0,t}-X+\int_{M_{0,t}}^\infty G(y;S_t,\tau)\mathrm{d}y & \Leftarrow M_{0,t}> X\end{cases} \tag{A.10}$$

The maximum price recorded until date T is given by:

$$M_{0,T}=\max\{M_{0,t},M_{t,T}\}$$

7.1. If $M_{0,t}\leq X$:

Let $y\in[X,\infty[$. As $M_{0,t}\leq X$ then $M_{0,t}\leq y$. So,

$$\mathcal{Q}(M_{0,T}\geq y)=\mathcal{Q}(M_{t,T}\geq y)\,. \tag{A.11}$$

Combining equations (A.9) and (A.11) we get

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T}-X)^+|\mathcal{F}_t\right] &= \int_X^\infty \mathcal{Q}(M_{t,T}\geq y|\mathcal{F}_t)\mathrm{d}y \\ &= \int_X^\infty G(y;S_t,\tau)\mathrm{d}y\,. \end{aligned} \tag{A.12}$$

7.2. If $M_{0,t}> X$:

Using equation (A.9), then

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T}-X)^+|\mathcal{F}_t\right] &= \int_X^\infty \mathcal{Q}(M_{0,T}\geq y|\mathcal{F}_t)\mathrm{d}y \\ &= \int_X^{M_{0,t}} \mathcal{Q}(M_{0,T}\geq y|\mathcal{F}_t)\mathrm{d}y + \int_{M_{0,t}}^\infty \mathcal{Q}(M_{0,T}\geq y|\mathcal{F}_t)\mathrm{d}y \\ &= \int_X^{M_{0,t}} 1\ \mathrm{d}y + \int_{M_{0,t}}^\infty \mathcal{Q}(M_{0,T}\geq y|\mathcal{F}_t)\mathrm{d}y\,.\end{aligned}$$

The last equality arises because $M_{0,T}\geq M_{0,t}$.

Let $y \in [M_{0,t}, \infty[$. As $M_{0,T} = \max\{M_{0,t}, M_{t,T}\}$, then $M_{0,T} \geq y \Leftrightarrow M_{t,T} \geq y$. Therefore,

$$\begin{aligned}\mathbb{E}_{\mathcal{Q}}\left[(M_{0,T} - X)^{+}|\mathcal{F}_t\right] &= M_{0,t} - X + \int_{M_{0,t}}^{\infty} \mathcal{Q}(M_{t,T} \geq y|\mathcal{F}_t)\mathrm{d}y \\ &= M_{0,t} - X + \int_{M_{0,t}}^{\infty} G(y; S_t, \tau)\mathrm{d}y\,. \qquad (A.13)\end{aligned}$$

Combining the equations (A.12) and (A.13), equation (A.10) arises.

8. Proof of equations (2.19) and (2.20).

The terminal payoff of a floating strike lookback put on asset S and with expiry date at time T is $LP_{fl}(T; S_T, M_{0,T}, T) = M_{0,T} - S_T$. The value at any time $t < T$ of the floating strike lookback call is:

$$\begin{aligned}LP_{fl}(t; S_t, M_{0,t}, T) &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\left[M_{0,T} - S_T|\mathcal{F}_t\right] \\ &= e^{-r\tau}\big(\mathbb{E}_{\mathcal{Q}}\left[M_{0,T} - M_{0,t}|\mathcal{F}_t\right]\big) + e^{-r\tau}\big(\mathbb{E}_{\mathcal{Q}}\left[M_{0,t} - S_T|\mathcal{F}_t\right]\big) \\ &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}(M_{0,T} - M_{0,t}|\mathcal{F}_t) + e^{-r\tau}M_{0,t} - e^{-r\tau}\mathbb{E}_{\mathcal{Q}}(S_T|\mathcal{F}_t) \\ &= e^{-r\tau}\mathbb{E}_{\mathcal{Q}}\big[(M_{0,T} - M_{0,t})^{+}|\mathcal{F}_t\big] + e^{-r\tau}M_{0,t} - e^{-r\tau}S_t e^{(r-q)\tau}.\end{aligned} \qquad (A.14)$$

This last equation was obtained due to the fact that $\mathbb{E}_{\mathcal{Q}}(S_T|\mathcal{F}_t) = S_t e^{(r-q)\tau}$ and $M_{0,T} \geq M_{0,t}$.

Combining equations (A.14) and (A.10) and replacing X for $M_{0,t}$,

$$LP_{fl}(t; S_t, M_{0,t}, T) = e^{-r\tau}\int_{M_{0,t}}^{\infty} G(y; S_t, \tau)\mathrm{d}y + e^{-r\tau}M_{0,t} - e^{-q\tau}S_t\,,$$

and equation (2.19) follows.

The terminal payoff of a fixed strike lookback call on asset S, with strike price K and expiry date at time T is $LC_{fx}(T; S_T, K, M_{0,T}, T) = (M_{0,T} - K)^{+}$.

The value at any time $t < T$ of the fixed strike lookback call is obtained replacing X for K in equation (A.10) and discounting, which yields equation (2.20).

□

Appendix B

Proof of Proposition 3

Following Davydov and Linetsky (2001),

$$\tau_{(L,U)} := \inf\{t \geq 0 : S_t \notin (L,U)\}\,,$$

and, therefore,

$$\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(L,U)}>T\}}(S_T-K)^+]\mathrm{d}T = \int_0^\infty e^{-\lambda T}\left[\int_L^U (Y-K)^+ p(T;S;Y)\mathrm{d}Y\right]\mathrm{d}T\,,$$

where $p(t;S;Y)$ is the transition density of the diffusion $\{S_t : t \geq 0\}$, starting at $S_0 = S$, with respect to the speed measure (see Borodin and Salminen, 1996, p.13). We are studying the case where the strike price (K) is less than the downside barrier (L), so we get $(Y-K)^+ = Y-K$ because the Y is between the downside barrier (L) and upper barrier (U). Using Fubbini's Theorem:

$$\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(L,U)}>T\}}(S_T-K)^+]\mathrm{d}T = \int_L^U (Y-K)\left[\int_0^\infty e^{-\lambda T}p(T;S;Y)\mathrm{d}T\right]\mathrm{d}Y\,.$$

The second integral on the right-hand side of the previous equation is the Green's function, $G_\lambda(S,Y)$, defined in Borodin and Salminen (1996, p.19). Therefore,

$$\int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(L,U)}>T\}}(S_T-K)^+]\mathrm{d}T = \int_L^U (Y-K)G_\lambda(S,Y)\mathrm{d}Y\,. \qquad \text{(B.1)}$$

The Green's function can be written as (see Ferreira (2009, p. 14-15) or Davydov and

Linetsky (2001, p. 962) for details):

$$G_\lambda(S,Y)\mathrm{d}Y = \begin{cases} \frac{m(Y)}{\omega_\lambda \Delta_\lambda(L,U)} \Delta_\lambda(L,S)\Delta_\lambda(Y,U) & \text{if } S \le Y \\ \frac{m(Y)}{\omega_\lambda \Delta_\lambda(L,U)} \Delta_\lambda(L,Y)\Delta_\lambda(S,U) & \text{if } S > Y \end{cases} . \tag{B.2}$$

As $S \in [L,U]$ (by definition of the double barrier option), then

$$\int_L^U (Y-K)G_\lambda(S,Y)dY = \int_L^S (Y-K)G_\lambda(S,Y)dY + \int_S^U (Y-K)G_\lambda(S,Y)dY \,. \tag{B.3}$$

When $Y \in [L,S]$ then $S > Y$, and if $Y \in [S,U]$ then $S \le Y$. Applying equation (B.2) to the first and second integrals on the right-hand side of equation (B.3), we get

$$\begin{aligned} \int_L^U (Y-K)G_\lambda(S,Y)dY &= \int_L^S (Y-K)\frac{m(Y)}{\omega_\lambda \Delta_\lambda(L,U)}\Delta_\lambda(L,Y)\Delta_\lambda(S,U)dY \\ &\quad + \int_S^U (Y-K)\frac{m(Y)}{\omega_\lambda \Delta_\lambda(L,U)}\Delta_\lambda(L,S)\Delta_\lambda(Y,U)dY \\ &= \frac{\Delta_\lambda(S,U)}{\omega_\lambda \Delta_\lambda(L,U)} \int_L^S (Y-K)m(Y)\Delta_\lambda(L,Y)dY \\ &\quad + \frac{\Delta_\lambda(L,S)}{\omega_\lambda \Delta_\lambda(L,U)} \int_S^U (Y-K)m(Y)\Delta_\lambda(Y,U)dY \,. \end{aligned} \tag{B.4}$$

Using equation (3.23), equation (B.4) can be written as

$$\begin{aligned} &\int_L^U (Y-K)G_\lambda(S,Y)dY \\ &= \frac{1}{\omega_\lambda \Delta_\lambda(L,U)} \Big[\Delta_\lambda(S,U) \int_L^S (Y-K)m(Y)\big(\phi_\lambda(L)\psi_\lambda(Y) - \psi_\lambda(L)\phi_\lambda(Y)\big)dY \\ &\qquad + \Delta_\lambda(L,S) \int_S^U (Y-K)m(Y)\big(\phi_\lambda(Y)\psi_\lambda(U) - \psi_\lambda(Y)\phi_\lambda(U)\big)dY\Big] \\ &= \frac{1}{\omega_\lambda \Delta_\lambda(L,U)} \Big\{\Delta_\lambda(S,U)\Big[\phi_\lambda(L) \int_L^S (Y-K)m(Y)\psi_\lambda(Y)dY \\ &\qquad - \psi_\lambda(L) \int_L^S (Y-K)m(Y)\phi_\lambda(Y)dY\Big] \\ &\qquad + \Delta_\lambda(L,S)\Big[\psi_\lambda(U) \int_S^U (Y-K)m(Y)\phi_\lambda(Y)dY \\ &\qquad - \phi_\lambda(U) \int_S^U (Y-K)m(Y)\psi_\lambda(Y))dY\Big]\Big\}. \end{aligned} \tag{B.5}$$

Using definitions (3.24) and (3.25) as well as equations (B.1) and (B.5), it follows that

$$
\begin{aligned}
& \int_0^\infty e^{-\lambda T}\mathbb{E}_{\mathcal{Q}}[\mathbb{1}_{\{\tau_{(L,U)}>T\}}(S_T-K)^+]\mathrm{d}T \\
= \quad & \frac{1}{\omega_\lambda \Delta_\lambda(L,U)}\Big[\Delta_\lambda(S,U)\big[\phi_\lambda(L)I_\lambda(K,L,S)-\psi_\lambda(L)J_\lambda(K,L,S)\big] \\
& \qquad +\Delta_\lambda(L,S)\big[\psi_\lambda(U)J_\lambda(K,S,U)-\phi_\lambda(U)I_\lambda(K,S,U)\big]\Big],
\end{aligned}
$$

which is exactly the equation (3.31). □

Appendix C

Proof of equations (4.6) - (4.9)

To prove equations (4.6) - (4.9) for the case that $r = q$ we determine the limit of these equations when $r \neq q$ as $r - q$ approaches zero.

First, we need to show that

$$\lim_{r-q\to 0}\big(-d_{bs1}(X,Y)\big) = \lim_{r-q\to 0}\big(d_{bs}(Y,X)\big) \tag{C.1}$$

Using equations (4.10) and (4.11), we get

$$\begin{aligned}
\lim_{r-q\to 0}\big(-d_{bs1}(X,Y)\big) &= \lim_{r-q\to 0}\left(-\frac{\ln(X/Y)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)\\
&= \frac{\ln(Y/X)-\frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\\
&= \lim_{r-q\to 0}\left(\frac{\ln(Y/X)+(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)\\
&= \lim_{r-q\to 0}\big(d_{bs}(Y,X)\big)\,.
\end{aligned}$$

Then we can proceed to the calculation of the limit for the lookback options as $r - q$ approaches zero.

1. Proof of equation (4.6).

$$
\begin{aligned}
& \lim_{r-q\to 0} LC_{fl}(t; S_t, m_{0,t}, T) \\
= & \lim_{r-q\to 0}\Bigg(S_t e^{-q\tau}\Phi[d_{bs1}(S_t, m_{0,t})] - m_{0,t}e^{-r\tau}\Phi[d_{bs}(S_t, m_{0,t})] \\
& +\frac{S_t}{\delta}\left\{ e^{-r\tau}\left(\frac{S_t}{m_{0,t}}\right)^{-\delta}\Phi[d_{bs}(m_{0,t}, S_t)] - e^{-q\tau}\Phi[-d_{bs1}(S_t, m_{0,t})]\right\}\Bigg) \\
= & S_t e^{-q\tau}\Phi[d_{bs1}(S_t, m_{0,t})] - m_{0,t}e^{-r\tau}\Phi[d_{bs}(S_t, m_{0,t})] \\
& +S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\left(\frac{\left(\frac{S_t}{m_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m_{0,t}, S_t)] - e^{(r-q)\tau}\Phi[-d_{bs1}(S_t, m_{0,t})]}{r-q}\right).
\end{aligned}
\tag{C.2}
$$

The last equation arises because $\delta = \frac{2(r-q)}{\sigma^2}$. When we replace the expression $r-q$ by zero in the last equation we obtain an indeterminate $0/0$ form. Applying the L'Hospital's rule, the indeterminate found becomes

$$
\begin{aligned}
& \lim_{r-q\to 0}\left(\frac{\left(\frac{S_t}{m_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m_{0,t}, S_t)] - e^{(r-q)\tau}\Phi[-d_{bs1}(S_t, m_{0,t})]}{r-q}\right) \\
= & \lim_{r-q\to 0}\Bigg\{ -\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m_{0,t}}\right)\left(\frac{S_t}{m_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m_{0,t}, S_t)] \\
& +\left(\frac{S_t}{m_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}} f[d_{bs}(m_{0,t}, S_t)]\frac{\sqrt{\tau}}{\sigma} - \tau e^{(r-q)\tau}\Phi[-d_{bs1}(S_t, m_{0,t})] \\
& -e^{(r-q)\tau} f[-d_{bs1}(S_t, m_{0,t})]\left(-\frac{\sqrt{\tau}}{\sigma}\right)\Bigg\},
\end{aligned}
\tag{C.3}
$$

where the last equation arises when we evaluate the derivatives of numerator and denominator with respect to $r-q$. Note that the derivative of $d_{bs}(Y, X)$ and $d_{bs1}(Y, X)$ are given by

$$
\begin{aligned}
\frac{\partial d_{bs}(Y, X)}{\partial(r-q)} &= \frac{\partial}{\partial(r-q)}\left(\frac{\ln(Y/X) + (r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) \\
&= \frac{\tau}{\sigma\sqrt{\tau}} \\
&= \frac{\sqrt{\tau}}{\sigma},
\end{aligned}
\tag{C.4}
$$

and

$$\frac{\partial d_{bs1}(Y,X)}{\partial(r-q)} = \frac{\partial}{\partial(r-q)}\left(\frac{\ln(Y/X)+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)$$
$$= \frac{\tau}{\sigma\sqrt{\tau}}$$
$$= \frac{\sqrt{\tau}}{\sigma}. \tag{C.5}$$

Using equation (C.1), and determining the limit of equation (C.3), we get

$$\lim_{r-q\to 0}\left(\frac{\left(\frac{S_t}{m_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m_{0,t},S_t)]-e^{(r-q)\tau}\Phi[-d_{bs1}(S_t,m_{0,t})]}{r-q}\right)$$
$$= -\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m_{0,t}}\right)\Phi[-d_{bs1}(S_t,m_{0,t})]+\frac{\sqrt{\tau}}{\sigma}f[-d_{bs1}(S_t,m_{0,t})]$$
$$-\tau\Phi[-d_{bs1}(S_t,m_{0,t})]+\frac{\sqrt{\tau}}{\sigma}f[-d_{bs1}(S_t,m_{0,t})]$$
$$= \frac{2\sqrt{\tau}}{\sigma}f[-d_{bs1}(S_t,m_{0,t})]-\Phi[-d_{bs1}(S_t,m_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m_{0,t}}\right)+\tau\right\}.$$

Combining the last equation with equation (C.2), we get

$$\lim_{r-q\to 0} LC_{fl}(t;S_t,m_{0,t},T)$$
$$= S_te^{-q\tau}\Phi[d_{bs1}(S_t,m_{0,t})]-m_{0,t}e^{-r\tau}\Phi[d_{bs}(S_t,m_{0,t})]$$
$$+S_te^{-r\tau}\frac{\sigma^2}{2}\left(\frac{2\sqrt{\tau}}{\sigma}f[-d_{bs1}(S_t,m_{0,t})]-\Phi[-d_{bs1}(S_t,m_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m_{0,t}}\right)+\tau\right\}\right)$$
$$= S_te^{-q\tau}\Phi[d_{bs1}(S_t,m_{0,t})]-m_{0,t}e^{-r\tau}\Phi[d_{bs}(S_t,m_{0,t})]$$
$$+S_te^{-r\tau}\sigma\sqrt{\tau}f[-d_{bs1}(S_t,m_{0,t})]$$
$$-S_te^{-r\tau}\frac{\sigma^2}{2}\Phi[-d_{bs1}(S_t,m_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m_{0,t}}\right)+\tau\right\},$$

which is the equation (4.6) when $r=q$.

2. Proof of equation (4.7).

$$
\begin{aligned}
&\lim_{r-q\to 0} LP_{fl}(t; S_t, K, M_{0,t}, T) \\
=\;& \lim_{r-q\to 0}\Bigg(M_{0,t}e^{-r\tau}\Phi[-d_{bs}(S_t, M_{0,t})] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, M_{0,t})] \\
&+\frac{S_t}{\delta}\left\{ e^{-q\tau}\Phi[d_{bs1}(S_t, M_{0,t})] - e^{-r\tau}\left(\frac{S_t}{M_{0,t}}\right)^{-\delta}\Phi[-d_{bs}(M_{0,t}, S_t)]\right\}\Bigg) \\
=\;& M_{0,t}e^{-r\tau}\Phi[-d_{bs}(S_t, M_{0,t})] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, M_{0,t})] \\
&+S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\left(\frac{e^{(r-q)\tau}\Phi[d_{bs1}(S_t, M_{0,t})] - \left(\frac{S_t}{M_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M_{0,t}, S_t)]}{r-q}\right),
\end{aligned}
\tag{C.6}
$$

using the definition of δ provided in equation (4.12). When we replace the expression $r-q$ by zero in the last equation we obtain an indeterminate $0/0$ form. Applying the L'Hospital's rule, the indeterminate found becomes

$$
\begin{aligned}
&\lim_{r-q\to 0}\left(\frac{e^{(r-q)\tau}\Phi[d_{bs1}(S_t, M_{0,t})] - \left(\frac{S_t}{M_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M_{0,t}, S_t)]}{r-q}\right) \\
=\;& \lim_{r-q\to 0}\Bigg\{ \tau e^{(r-q)\tau}\Phi[d_{bs1}(S_t, M_{0,t})] + e^{(r-q)\tau} f[d_{bs1}(S_t, M_{0,t})]\left(\frac{\sqrt{\tau}}{\sigma}\right) \\
&-\left(-\frac{2}{\sigma^2}\right)\ln\left(\frac{S_t}{M_{0,t}}\right)\left(\frac{S_t}{M_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M_{0,t}, S_t)] \\
&-\left(\frac{S_t}{M_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}} f[-d_{bs}(M_{0,t}, S_t)]\left(-\frac{\sqrt{\tau}}{\sigma}\right)\Bigg\},
\end{aligned}
\tag{C.7}
$$

where the last equation arises when we evaluate the derivatives of numerator and denominator with respect to $r-q$. Note that the derivative of $d_{bs}(Y, X)$ and $d_{bs1}(Y, X)$ are given by equations (C.4) and (C.5).

Using equation (C.1), the equation (C.7) becomes,

$$\begin{aligned}
&\lim_{r-q\to 0}\left(\frac{e^{(r-q)\tau}\Phi[d_{bs1}(S_t,M_{0,t})]-\left(\frac{S_t}{M_{0,t}}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M_{0,t},S_t)]}{r-q}\right)\\
= &\ \tau\Phi[d_{bs1}(S_t,M_{0,t})]+\frac{\sqrt{\tau}}{\sigma}f[d_{bs1}(S_t,M_{0,t})]\\
&+\frac{2}{\sigma^2}\ln\left(\frac{S_t}{M_{0,t}}\right)\Phi[d_{bs1}(S_t,M_{0,t})]+\frac{\sqrt{\tau}}{\sigma}f[d_{bs1}(S_t,M_{0,t})]\\
= &\ \frac{2\sqrt{\tau}}{\sigma}f[d_{bs1}(S_t,M_{0,t})]+\Phi[d_{bs1}(S_t,M_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{M_{0,t}}\right)+\tau\right\}.
\end{aligned}$$

Combining the last equation with equation (C.6), we get

$$\begin{aligned}
&\lim_{r-q\to 0} LP_{fl}(t;S_t,K,M_{0,t},T)\\
= &\ M_{0,t}e^{-r\tau}\Phi[-d_{bs}(S_t,M_{0,t})]-S_te^{-q\tau}\Phi[-d_{bs1}(S_t,M_{0,t})]\\
&+S_te^{-r\tau}\frac{\sigma^2}{2}\left(\frac{2\sqrt{\tau}}{\sigma}f[d_{bs1}(S_t,M_{0,t})]+\Phi[d_{bs1}(S_t,M_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{M_{0,t}}\right)+\tau\right\}\right)\\
= &\ M_{0,t}e^{-r\tau}\Phi[-d_{bs}(S_t,M_{0,t})]-S_te^{-q\tau}\Phi[-d_{bs1}(S_t,M_{0,t})]\\
&+S_te^{-r\tau}\sigma\sqrt{\tau}f[d_{bs1}(S_t,M_{0,t})]\\
&+S_te^{-r\tau}\frac{\sigma^2}{2}\Phi[d_{bs1}(S_t,M_{0,t})]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{M_{0,t}}\right)+\tau\right\},
\end{aligned}\tag{C.8}$$

which is equation (4.7) for the case $r=q$.

3. Proof of equation (4.8).

   Using the same arguments applied to the proofs of equations (4.6) and (4.7), we obtain a similar result for the equation (4.8).

$$\begin{aligned}
&\lim_{r-q\to 0} LC_{fx}(t;S_t,K,M_{0,t},T)\\
= &\lim_{r-q\to 0}\Bigg(e^{-r\tau}\left(M_{0,t}-K\right)^{+}+S_te^{-q\tau}\Phi[d_{bs1}(S_t,\max\{M_{0,t},K\})]\\
&-\max\{M_{0,t},K\}e^{-r\tau}\Phi[d_{bs}(S_t,\max\{M_{0,t},K\})]\\
&+\frac{S_t}{\delta}\Bigg\{e^{-q\tau}\Phi[d_{bs1}(S_t,\max\{M_{0,t},K\})]\\
&-e^{-r\tau}\left(\frac{S_t}{\max\{M_{0,t},K\}}\right)^{-\delta}\Phi[-d_{bs}(\max\{M_{0,t},K\},S_t)]\Bigg\}\Bigg)
\end{aligned}$$

To simplify the expression, let $M = \max\{M_{0,t}, K\}$. Then

$$
\begin{aligned}
& \lim_{r-q\to 0} LC_{fx}(t; S_t, K, M_{0,t}, T) \\
= \;& e^{-r\tau}\big(M_{0,t} - K\big)^{+} + S_t e^{-q\tau}\Phi[d_{bs1}(S_t, M)] - M e^{-r\tau}\Phi[d_{bs}(S_t, M)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\left(\frac{e^{(r-q)\tau}\Phi[d_{bs1}(S_t, M)] - \left(\frac{S_t}{M}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M, S_t)]}{r-q}\right) \\
= \;& e^{-r\tau}\big(M_{0,t} - K\big)^{+} + S_t e^{-q\tau}\Phi[d_{bs1}(S_t, M)] - M e^{-r\tau}\Phi[d_{bs}(S_t, M)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\Bigg(\tau e^{(r-q)\tau}\Phi[d_{bs1}(S_t, M)] + e^{(r-q)\tau} f[d_{bs1}(S_t, M)]\left(\frac{\sqrt{\tau}}{\sigma}\right) \\
& - \left(-\frac{2}{\sigma^2}\right)\ln\left(\frac{S_t}{M}\right)\left(\frac{S_t}{M}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[-d_{bs}(M, S_t)] \\
& - \left(\frac{S_t}{M}\right)^{-\frac{2(r-q)}{\sigma^2}} f[-d_{bs}(M, S_t)]\left(-\frac{\sqrt{\tau}}{\sigma}\right)\Bigg) \\
= \;& e^{-r\tau}\big(M_{0,t} - K\big)^{+} + S_t e^{-q\tau}\Phi[d_{bs1}(S_t, M)] - M e^{-r\tau}\Phi[d_{bs}(S_t, M)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\Bigg(\tau\Phi[d_{bs1}(S_t, M)] + \frac{\sqrt{\tau}}{\sigma} f[d_{bs1}(S_t, M)] + \frac{2}{\sigma^2}\ln\left(\frac{S_t}{M}\right)\Phi[d_{bs1}(S_t, M)] \\
& + \frac{\sqrt{\tau}}{\sigma} f[d_{bs1}(S_t, M)]\Bigg) \\
= \;& e^{-r\tau}\big(M_{0,t} - K\big)^{+} + S_t e^{-q\tau}\Phi[d_{bs1}(S_t, M)] - M e^{-r\tau}\Phi[d_{bs}(S_t, M)] \\
& + S_t e^{-r\tau}\sigma\sqrt{(\tau)} f[d_{bs1}(S_t, M)] + S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi[d_{bs1}(S_t, M)]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{M}\right) + \tau\right\},
\end{aligned}
$$

which is the equation (4.8), when $r = q$.

4. Proof of equation (4.9). Using the same arguments applied to the proofs of equations (4.6) - (4.8), we obtain a similar result for the equation (4.9).

$$
\begin{aligned}
& \lim_{r-q\to 0} LP_{fx}(t; S_t, K, M_{0,t}, T) \\
= \;& \lim_{r-q\to 0}\Bigg(e^{-r\tau}\big(K - m_{0,t}\big)^{+} + \min\{m_{0,t}, K\} e^{-r\tau}\Phi[-d_{bs}(S_t, \min\{m_{0,t}, K\})] \\
& - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, \min\{m_{0,t}, K\})] + \frac{S_t}{\delta}\Bigg\{-e^{-q\tau}\Phi[-d_{bs1}(S_t, \min\{m_{0,t}, K\})] \\
& + e^{-r\tau}\left(\frac{S_t}{\min\{m_{0,t}, K\}}\right)^{-\delta}\Phi[d_{bs}(\min\{m_{0,t}, K\}, S_t)]\Bigg\}\Bigg)
\end{aligned}
$$

To simplify the expression, let $m = \min\{m_{0,t}, K\}$. Therefore,

$$
\begin{aligned}
& \lim_{r-q\to 0} LP_{fx}(t; S_t, K, M_{0,t}, T) \\
= \; & e^{-r\tau}\big(K - m_{0,t}\big)^+ + me^{-r\tau}\Phi[-d_{bs}(S_t, m)] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, m)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\left(\frac{-e^{(r-q)\tau}\Phi[-d_{bs1}(S_t, m)] + \left(\frac{S_t}{m}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m, S_t)]}{r-q}\right) \\
= \; & e^{-r\tau}\big(K - m_{0,t}\big)^+ + me^{-r\tau}\Phi[-d_{bs}(S_t, m)] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, m)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\lim_{r-q\to 0}\Bigg\{ - \tau e^{(r-q)\tau}\Phi[-d_{bs1}(S_t, m)] \\
& - e^{(r-q)\tau} f[-d_{bs1}(S_t, m)]\left(- \frac{\sqrt{\tau}}{\sigma}\right) \\
& + \left(- \frac{2}{\sigma^2}\right)\ln\left(\frac{S_t}{m}\right)\left(\frac{S_t}{m}\right)^{-\frac{2(r-q)}{\sigma^2}}\Phi[d_{bs}(m, S_t)] \\
& + \left(\frac{S_t}{m}\right)^{-\frac{2(r-q)}{\sigma^2}} f[d_{bs}(m, S_t)]\left(\frac{\sqrt{\tau}}{\sigma}\right)\Bigg\} \\
= \; & e^{-r\tau}\big(K - m_{0,t}\big)^+ + me^{-r\tau}\Phi[-d_{bs}(S_t, m)] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, m)] \\
& + S_t e^{-r\tau}\frac{\sigma^2}{2}\Bigg\{ - \tau\Phi[-d_{bs1}(S_t, m)] + \frac{\sqrt{\tau}}{\sigma} f[-d_{bs1}(S_t, m)] \\
& - \frac{2}{\sigma^2}\ln\left(\frac{S_t}{m}\right)\Phi[-d_{bs1}(S_t, m)] + \frac{\sqrt{\tau}}{\sigma} f[-d_{bs1}(S_t, m)]\Bigg\} \\
= \; & e^{-r\tau}\big(K - m_{0,t}\big)^+ + me^{-r\tau}\Phi[-d_{bs}(S_t, m)] - S_t e^{-q\tau}\Phi[-d_{bs1}(S_t, m)] \\
& + S_t e^{-r\tau}\sigma\sqrt{\tau} f[-d_{bs1}(S_t, m)] - S_t e^{-r\tau}\frac{\sigma^2}{2}\Phi[-d_{bs}(S_t, m)]\left\{\frac{2}{\sigma^2}\ln\left(\frac{S_t}{m}\right) + \tau\right\},
\end{aligned}
$$

which is the equation (4.9) for the case $r = q$.

$\square$

Appendix D

Proof of Proposition 5

To prove Proposition 5 we will use equations (4.2), (4.3), (4.4), (4.5), (4.17) and (4.18):

$$\begin{aligned}
\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) &= \lim_{U\to\infty} \frac{U^{\gamma_-}}{U^{\gamma_+}} \frac{2}{\sigma^2} \left(\frac{U^{1-\gamma_-}}{1-\gamma_-} + K\frac{U^{-\gamma_-}}{\gamma_-} - \frac{S^{1-\gamma_-}}{1-\gamma_-} - K\frac{S^{-\gamma_-}}{\gamma_-} \right) \\
&= \frac{2}{\sigma^2} \lim_{U\to\infty} U^{1-\gamma_+} \left(\frac{1}{1-\gamma_-} + K\frac{U^{-1}}{\gamma_-} \right) \\
&\quad - \frac{2}{\sigma^2} \left(\frac{S^{1-\gamma_-}}{1-\gamma_-} + K\frac{S^{-\gamma_-}}{\gamma_-} \right) \lim_{U\to\infty} U^{\gamma_- - \gamma_+} \,. \qquad (D.1)
\end{aligned}$$

Using the definitions of γ_- and γ_+ provided in equation (4.4), we get

$$1 - \gamma_+ = 1 + \gamma - \sqrt{\gamma^2 + \frac{2\lambda}{\sigma^2}}\,, \qquad (D.2)$$

i.e.

$$1 - \gamma_+ < 0\,, \qquad (D.3)$$

as long as λ is large enough for inequality (4.19) to be verified.

And

$$\begin{aligned}
\gamma_- - \gamma_+ &= -\gamma - \sqrt{\gamma^2 + \frac{2\lambda}{\sigma^2}} + \gamma - \sqrt{\gamma^2 + \frac{2\lambda}{\sigma^2}} \\
&= -2\sqrt{\gamma^2 + \frac{2\lambda}{\sigma^2}}\,, \qquad (D.4)
\end{aligned}$$

i.e.

$$\gamma_- - \gamma_+ < 0\,. \qquad (D.5)$$

Using equations (D.3) and (D.5), the equation (D.1) becomes

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)=0\,. \tag{D.6}$$

For the integral $J_\lambda(K,S,U)$, equation (5.15) yields:

$$\begin{aligned}\lim_{U\to\infty}J_\lambda(K,S,U) &= \lim_{U\to\infty}\frac{2}{\sigma^2}\left(\frac{U^{1-\gamma_+}}{1-\gamma_+}+K\frac{U^{-\gamma_+}}{\gamma_+}-\frac{S^{1-\gamma_+}}{1-\gamma_+}-K\frac{S^{-\gamma_+}}{\gamma_+}\right)\\ &= \frac{2}{\sigma^2}\lim_{U\to\infty}\left(\frac{U^{1-\gamma_+}}{1-\gamma_+}+K\frac{U^{-\gamma_+}}{\gamma_+}\right)-\frac{2}{\sigma^2}\left(\frac{S^{1-\gamma_+}}{1-\gamma_+}+K\frac{S^{-\gamma_+}}{\gamma_+}\right).\end{aligned} \tag{D.7}$$

Using the definition of γ_+ provided in equation (4.4), it follows that

$$-\gamma_+=\gamma-\sqrt{\gamma^2+\frac{\lambda}{\sigma^2}}\,. \tag{D.8}$$

As $\gamma\le\sqrt{\gamma^2}$ and $\sqrt{\gamma^2}<\sqrt{\gamma^2+\frac{2\lambda}{\sigma^2}}$ (because $\lambda>0$), the expression $-\gamma_+$ in equation (D.8) is negative.

Using equations (D.8) and (D.3), equation (D.7) becomes

$$\lim_{U\to\infty}J_\lambda(K,S,U)=-\frac{2}{\sigma^2}\left(\frac{S^{1-\gamma_+}}{1-\gamma_+}+K\frac{S^{-\gamma_+}}{\gamma_+}\right)\,. \tag{D.9}$$

□

Appendix E

Proof of Proposition 7

To prove Proposition 7 we will use equations (5.4)-(5.11), (5.14) and (5.15).

- If $\mu > 0$ and $\beta < 0$

$$
\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \;& \lim_{U\to\infty} \frac{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} W_{k,m}(x(U))}{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} M_{k,m}(x(U))} \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{U^{1/2}}{2m+1} \exp\left(\frac{x(U)}{2}\right) M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \right. \\
& - \frac{2mKU^{-1/2}}{m-k-\frac{1}{2}} \exp\left(\frac{x(U)}{2}\right) M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
& - \frac{S^{1/2}}{2m+1} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\
& \left. + \frac{2mKS^{-1/2}}{m-k-\frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \right] \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{W_{k,m}(x(U))}{M_{k,m}(x(U))} \exp\left(\frac{x(U)}{2}\right) U^{1/2} \left[\frac{1}{2m+1} M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \right. \right. \\
& \left. \left. - \frac{2mKU^{-1}}{m-k-\frac{1}{2}} M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \right] \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{S^{1/2}}{2m+1} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. - \frac{2mKS^{-1/2}}{m-k-\frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \right] \lim_{U\to\infty} \left(\frac{W_{k,m}(x(U))}{M_{k,m}(x(U))} \right) .
\end{aligned}
$$

Using equations (13.1.32) and (13.1.33) of Abramowitz and Stegun (1972, p.505) the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}\right.\\
&\exp\left(\frac{x(U)}{2}\right)U^{1/2}\left[\frac{1}{2m+1}e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{1+m}\mathcal{M}\left(\frac{1}{2}+m-k,2+2m,x(U)\right)\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{m}\mathcal{M}\left(-\frac{1}{2}+m-k,2m,x(U)\right)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{2mKS^{-1/2}}{m-k-\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}\ ,
\end{aligned}
$$

where $\mathcal{U}(a,b,x)$ and $\mathcal{M}(a,b,x)$ are the Kummer's functions.

Given that $\lim_{U\to\infty}x(U)=+\infty$ (because $\beta<0$) and using equations (13.1.4) and (13.1.8) of Abramowitz and Stegun (1972, p.504), it follows that

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}(x(U))^{-(\frac{1}{2}+m-k)}}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}(x(U))^{\frac{1}{2}+m-k-1-2m}}\right.\\
&\exp\left(\frac{x(U)}{2}\right)U^{1/2}\left[\frac{1}{2m+1}e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{1+m}\frac{\Gamma(2m+2)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}\big(x(U)\big)^{-m-\frac{3}{2}-k}\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{m}\frac{\Gamma(2m)}{\Gamma(m-\frac{1}{2}-k)}e^{x(U)}\big(x(U)\big)^{m-\frac{1}{2}-k-2m}\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{2mKS^{-1/2}}{m-k-\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}(x(U))^{-(\frac{1}{2}+m-k)}}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}(x(U))^{\frac{1}{2}+m-k-1-2m}}\ .
\end{aligned}
$$

Simplifying the right-hand side of the last equation, we get

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{\big(x(U)\big)^k e^{-\frac{1}{2}x(U)}}{\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-k}e^{\frac{1}{2}x(U)}}\exp\left(\frac{x(U)}{2}\right)U^{1/2}\right.\\
&\left[\frac{1}{2m+1}\frac{\Gamma(2m+2)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-k-\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(m-\frac{1}{2}-k)}\big(x(U)\big)^{-k-\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{2mKS^{-1/2}}{m-k-\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\lim_{U\to\infty}\frac{\big(x(U)\big)^k e^{-\frac{1}{2}x(U)}}{\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-k}e^{\frac{1}{2}x(U)}}\\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\lim_{U\to\infty}\left\{\big(x(U)\big)^{k-\frac{1}{2}}U^{1/2}\left[\frac{1}{2m+1}\frac{\Gamma(2m+2)}{\Gamma(\frac{1}{2}+m-k)}\right.\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(m-\frac{1}{2}-k)}\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{2mKS^{-1/2}}{m-k-\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\lim_{U\to\infty}\frac{\big(x(U)\big)^{2k}}{e^{x(U)}}\\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{k-\frac{1}{2}}U^{1/2}\left[\frac{1}{2m+1}\frac{\Gamma(2m+2)}{\Gamma(\frac{1}{2}+m-k)}\right.\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(m-\frac{1}{2}-k)}\right]\right\}.
\end{aligned}
$$

The last equation arises from the definition of $x(U)$ in equation (5.6). Therefore, the last equation can be written as

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= \;&\frac{1}{\delta\sqrt{|\beta\mu|}}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k-\frac{1}{2}}U^{-2\beta(k-\frac{1}{2})+\frac{1}{2}}\left[\frac{1}{2m+1}\frac{\Gamma(2m+2)}{\Gamma(\frac{1}{2}+m-k)}\right.\right.\\
&\left.\left.-\frac{2mKU^{-1}}{m-k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(m-\frac{1}{2}-k)}\right]\right\}. \qquad \text{(E.1)}
\end{aligned}
$$

Assuming that $\lambda > 0$ is such that $-2\beta(-\frac{1}{2}-\frac{1}{4\beta}-\frac{\lambda}{2|\mu\beta|}-\frac{1}{2})+\frac{1}{2}<0$, then equation (E.1) is equal to zero.

For the integral $J_\lambda(K,S,U)$, equation (5.15) yields:

$$
\begin{aligned}
&\lim_{U\to\infty}J_\lambda(K,S,U)\\
= \;&\lim_{U\to\infty}\left\{\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{U^{1/2}}{k+m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\right.\right.\\
&-\frac{KU^{-1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\\
&-\frac{S^{1/2}}{k+m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\
&\left.\left.+\frac{KS^{-1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\right\}\\
= \;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\exp\left(\frac{x(U)}{2}\right)\left[\frac{U^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\right.\right.\\
&\left.\left.-\frac{KU^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}e^{\frac{x(S)}{2}}\left[\frac{S^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)-\frac{KS^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned}
$$

Using equation (13.1.33) of Abramowitz and Stegun (1972, p.505) the last equation

becomes

$$
\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\exp\left(\frac{x(U)}{2}\right)\Bigg[\frac{U^{1/2}}{k+m+\frac{1}{2}}\exp\Big(-\frac{x(U)}{2}\Big)\big(x(U)\big)^{1+m}\\
&\mathcal{U}\Big(\frac{1}{2}+m-k,2+2m,x(U)\Big)\\
&-\frac{KU^{-1/2}}{k-m+\frac{1}{2}}\exp\Big(-\frac{x(U)}{2}\Big)\big(x(U)\big)^{m}\mathcal{U}\Big(-\frac{1}{2}+m-k,2m,x(U)\Big)\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}e^{\frac{x(S)}{2}}\left[\frac{S^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)-\frac{KS^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right],
\end{aligned}
$$

where $\mathcal{U}(a,b,x)$ is the Kummer's function.
Given that $\lim_{U\to\infty} x(U) = +\infty$ (because $\beta < 0$) and using equation (13.1.8) of Abramowitz and Stegun (1972, p.504), it follows that

$$
\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{U^{1/2}}{k+m+\frac{1}{2}}\big(x(U)\big)^{1+m}\big(x(U)\big)^{-\frac{1}{2}-m+k}\\
&-\frac{KU^{-1/2}}{k-m+\frac{1}{2}}\big(x(U)\big)^{m}\big(x(U)\big)^{\frac{1}{2}-m+k}\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}e^{\frac{x(S)}{2}}\left[\frac{S^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)-\frac{KS^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\big(x(U)\big)^{k+\frac{1}{2}}U^{1/2}\left[\frac{1}{k+m+\frac{1}{2}}-\frac{KU^{-1}}{k-m+\frac{1}{2}}\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}e^{\frac{x(S)}{2}}\left[\frac{S^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)-\frac{KS^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{k+\frac{1}{2}}U^{1/2}\left[\frac{1}{k+m+\frac{1}{2}}-\frac{KU^{-1}}{k-m+\frac{1}{2}}\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}e^{\frac{x(S)}{2}}\left[\frac{S^{1/2}}{k+m+\frac{1}{2}}W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)-\frac{KS^{-1/2}}{k-m+\frac{1}{2}}W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned}
$$

Simplifying the right-hand side of the last equation, we get

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
&= \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k+\frac{1}{2}} U^{-2\beta(k+\frac{1}{2})+\frac{1}{2}} \left[\frac{1}{k+m+\frac{1}{2}} - \frac{KU^{-1}}{k-m+\frac{1}{2}}\right]\right\}\\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}} e^{\frac{x(S)}{2}} \left[\frac{S^{1/2}}{k+m+\frac{1}{2}} W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) - \frac{KS^{-1/2}}{k-m+\frac{1}{2}} W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned} \tag{E.2}$$

Using the definition of k provided in equation (5.10) the expression $-2\beta(k+\frac{1}{2})+\frac{1}{2}$ can be written as

$$\begin{aligned}
-2\beta\left(k+\frac{1}{2}\right)+\frac{1}{2} &= -2\beta\left(-\frac{1}{2}-\frac{1}{4\beta}-\frac{\lambda}{2|\mu\beta|}+\frac{1}{2}\right)+\frac{1}{2}\\
&= \frac{1}{2}-\frac{\lambda}{|\mu|}+\frac{1}{2}\\
&= 1-\frac{\lambda}{|\mu|}
\end{aligned} \tag{E.3}$$

For $\lambda > 0$ large enough such that $1-\frac{\lambda}{|\mu|} < 0$, equation (E.3) is negative and equation (E.2) becomes

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
&= -\frac{1}{\delta\sqrt{|\beta\mu|}} e^{\frac{x(S)}{2}} \left[\frac{S^{1/2}}{k+m+\frac{1}{2}} W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) - \frac{KS^{-1/2}}{k-m+\frac{1}{2}} W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned}$$

- If $\mu < 0$ and $\beta < 0$, equations (5.4), (5.5) and (5.14) imply that

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U)\\
&= \lim_{U\to\infty} \frac{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} W_{k,m}(x(U))}{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} M_{k,m}(x(U))} \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{U^{1/2}}{2m+1} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\right.\\
&\quad +\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\\
&\quad -\frac{S^{1/2}}{2m+1} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\
&\quad \left. -\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right],
\end{aligned}$$

i.e.,

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= &\lim_{U\to\infty}\left\{\frac{W_{k,m}(x(U))}{M_{k,m}(x(U))}\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{U^{1/2}}{2m+1}\exp\left(-\frac{x(U)}{2}\right)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\right.\right.\\
&\left.\left.+\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}}\exp\left(-\frac{x(U)}{2}\right)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.+\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\lim_{U\to\infty}\frac{W_{k,m}(x(U))}{M_{k,m}(x(U))}\ .
\end{aligned}
$$

Using equations (13.1.32) and (13.1.33) of Abramowitz and Stegun (1972, p.505) the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= &\lim_{U\to\infty}\left\{\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}\frac{1}{\delta\sqrt{|\beta\mu|}}\right.\\
&\left[\frac{U^{1/2}}{2m+1}\exp\left(-\frac{x(U)}{2}\right)e^{-\frac{1}{2}x(U)}(x(U))^{1+m}\mathcal{M}\left(\frac{3}{2}+m-k,2+2m,x(U)\right)\right.\\
&\left.\left.+\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}}\exp\left(-\frac{x(U)}{2}\right)e^{-\frac{1}{2}x(U)}(x(U))^{m}\mathcal{M}\left(\frac{1}{2}+m-k,2m,x(U)\right)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{2m+1}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.+\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}\ ,
\end{aligned}
$$

where $\mathcal{M}(a,b,x)$ and $\mathcal{U}(a,b,x)$ are the Kummer's functions.
Given that $\lim_{U\to\infty}x(U)=+\infty$ (because $\beta<0$) and using equations (13.1.4) and

(13.1.8) of Abramowitz and Stegun (1972, p.504), it follows that

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
&=\lim_{U\to\infty}\Bigg\{\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}(x(U))^{-\frac{1}{2}-m+k}}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}(x(U))^{\frac{1}{2}+m-k-1-2m}}\frac{1}{\delta\sqrt{|\beta\mu|}}\\
&\Bigg[\frac{U^{1/2}}{2m+1}e^{-x(U)}(x(U))^{1+m}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}e^{x(U)}\big(x(U)\big)^{\frac{3}{2}+m-k-2-2m}\\
&+\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}}e^{-x(U)}(x(U))^{m}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}\big(x(U)\big)^{\frac{1}{2}+m-k-2m}\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{2m+1}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\
&+\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Bigg]\\
&\lim_{U\to\infty}\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}(x(U))^{-\frac{1}{2}-m+k}}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}e^{x(U)}(x(U))^{\frac{1}{2}+m-k-1-2m}}\\
&=\lim_{U\to\infty}\Bigg\{\frac{\Gamma(\frac{1}{2}+m-k)\big(x(U)\big)^{k}e^{-\frac{x(U)}{2}}}{\Gamma(1+2m)\big(x(U)\big)^{-k}e^{\frac{x(U)}{2}}}\frac{1}{\delta\sqrt{|\beta\mu|}}\\
&\Bigg[\frac{U^{1/2}}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}\big(x(U)\big)^{-k+\frac{1}{2}}+\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-k+\frac{1}{2}}\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{2m+1}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\
&+\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Bigg]\lim_{U\to\infty}\frac{\Gamma(\frac{1}{2}+m-k)\big(x(U)\big)^{k}e^{-\frac{x(U)}{2}}}{\Gamma(1+2m)\big(x(U)\big)^{-k}e^{\frac{x(U)}{2}}}\\
&=\lim_{U\to\infty}\Bigg\{\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\frac{1}{\delta\sqrt{|\beta\mu|}}\big(x(U)\big)^{k+\frac{1}{2}}e^{-x(U)}\Bigg[\frac{U^{1/2}}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}\\
&+\frac{2mKU^{-1/2}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{2m+1}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\
&+\frac{2mKS^{-1/2}}{m+k-\frac{1}{2}}\exp\Big(-\frac{x(S)}{2}\Big)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Bigg]\\
&\lim_{U\to\infty}\Bigg(\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\big(x(U)\big)^{2k}e^{-x(U)}\Bigg).
\end{aligned}
$$

Using the definition of $x(U)$ provided in equation (5.6), the last equation becomes

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{k+\frac{1}{2}}e^{-x(U)}U^{1/2}\right.\\
&\left.\left[\frac{1}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}+\frac{2mKU^{-1}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\right]\right\}\\
=\;&\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k+\frac{1}{2}}e^{-x(U)}U^{-2\beta(k+\frac{1}{2})+\frac{1}{2}}\right.\\
&\left.\left[\frac{1}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}+\frac{2mKU^{-1}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\right]\right\}.
\end{aligned}\tag{E.4}$$

Using the definition of k provided in equation (5.10), the expression $-2\beta(k+\frac{1}{2})+\frac{1}{2}$ can be written as

$$\begin{aligned}
-2\beta\left(k+\frac{1}{2}\right)+\frac{1}{2} &= -2\beta\left(\frac{1}{2}+\frac{1}{4\beta}-\frac{\lambda}{2|\mu\beta|}+\frac{1}{2}\right)+\frac{1}{2}\\
&= -2\beta-\frac{1}{2}-\frac{\lambda}{|\mu|}+\frac{1}{2}\\
&= -2\beta-\frac{\lambda}{|\mu|}\,.
\end{aligned}\tag{E.5}$$

Replacing equation (E.5) in equation (E.4), we get

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k+\frac{1}{2}}e^{-x(U)}\frac{\delta^2|\beta|}{|\mu|}\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}U^{-\frac{\lambda}{|\mu|}}\right.\\
&\left.\left[\frac{1}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}+\frac{2mKU^{-1}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\right]\right\}\\
=\;&\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k+\frac{1}{2}}\frac{\delta^2|\beta|}{|\mu|}\frac{x(U)}{e^{x(U)}}U^{-\frac{\lambda}{|\mu|}}\right.\\
&\left.\left[\frac{1}{2m+1}\frac{\Gamma(2+2m)}{\Gamma(\frac{3}{2}+m-k)}+\frac{2mKU^{-1}}{m+k-\frac{1}{2}}\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\right]\right\}\\
=\;&0\,.
\end{aligned}$$

For the integral $J_\lambda(K,S,U)$, equation (5.15) yields:

$$\begin{aligned}\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[U^{1/2}\exp\Big(-\frac{x(U)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\\ &\quad +KU^{-1/2}\exp\Big(-\frac{x(U)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\\ &\quad -S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &\quad -KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big]\Bigg\}.\end{aligned}$$

Simplifying the right-hand side of the last equation, it follows that

$$\begin{aligned}\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}\exp\Big(-\frac{x(U)}{2}\Big)\Big[W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\\ &\quad +KU^{-1}W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\Bigg\}\\ &\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big].\end{aligned}$$

Using equation (13.1.33) of Abramowitz and Stegun (1972, p.505) the last equation becomes

$$\begin{aligned}&\lim_{U\to\infty} J_\lambda(K,S,U) =\\ =\;& \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}\exp\Big(-\frac{x(U)}{2}\Big)\Bigg[\exp\Big(-\frac{x(U)}{2}\Big)\big(x(U)\big)^{1+m}\\ &\mathcal{U}\Big(\frac{3}{2}+m-k,2+2m,x(U)\Big)\\ &+KU^{-1}\exp\Big(-\frac{x(U)}{2}\Big)\big(x(U)\big)^{m}\mathcal{U}\Big(\frac{1}{2}+m-k,2m,x(U)\Big)\Bigg]\Bigg\}\\ &-\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &+KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big],\end{aligned}$$

where $\mathcal{U}(a,b,x)$ is the Kummer's function.
Given that $\lim_{U\to\infty} x(U) = +\infty$ (because $\beta < 0$) and using equation (13.1.8) of Abramowitz and Stegun (1972, p.504), we get

$$\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}e^{-x(U)}\Big[\big(x(U)\big)^{1+m}\big(x(U)\big)^{-\frac{3}{2}-m+k} \\
&\quad +KU^{-1}\big(x(U)\big)^{m}\big(x(U)\big)^{-\frac{1}{2}-m+k}\Bigg\} \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\
&\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big] \\
&= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}e^{-x(U)}\Big[\big(x(U)\big)^{k-\frac{1}{2}}+KU^{-1}\big(x(U)\big)^{k-\frac{1}{2}}\Bigg\} \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\
&\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big] \\
&= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}e^{-x(U)}\big(x(U)\big)^{k-\frac{1}{2}}\Big[1+KU^{-1}\Big]\Bigg\} \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\
&\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big].
\end{aligned}$$

Using the definition of $x(U)$ provided in equation (5.6) the last equation becomes

$$\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\Bigg\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{1/2}e^{-x(U)}\Big(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\Big)^{k-\frac{1}{2}}\Big[1+KU^{-1}\Big]\Bigg\} \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \\
&\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big],
\end{aligned}$$

i.e.,

$$\begin{aligned}\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\left\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{-2\beta(k-\frac{1}{2})+\frac{1}{2}}e^{-x(U)}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k-\frac{1}{2}}\Big[1+KU^{-1}\Big]\right\}\\ &\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big]. \qquad \text{(E.6)}\end{aligned}$$

Using the definition of k provided in equation (5.10) the expression $-2\beta(k-\frac{1}{2})+\frac{1}{2}$ can be written as

$$\begin{aligned}-2\beta\left(k-\frac{1}{2}\right)+\frac{1}{2} &= -2\beta\left(\frac{1}{2}+\frac{1}{4\beta}-\frac{\lambda}{2|\mu\beta|}-\frac{1}{2}\right)+\frac{1}{2}\\ &= -\frac{1}{2}-\frac{\lambda}{|\mu|}+\frac{1}{2}\\ &= -\frac{\lambda}{|\mu|}. \qquad \text{(E.7)}\end{aligned}$$

Replacing equation (E.7) into equation (E.6), it follows that

$$\begin{aligned}\lim_{U\to\infty} J_\lambda(K,S,U) &= \lim_{U\to\infty}\left\{\frac{1}{\delta\sqrt{|\beta\mu|}}U^{-\frac{\lambda}{|\mu|}}e^{-x(U)}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{k-\frac{1}{2}}\Big[1+KU^{-1}\Big]\right\}\\ &\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big]\\ &= -\frac{1}{\delta\sqrt{|\beta\mu|}}\Big[S^{1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\\ &\quad +KS^{-1/2}\exp\Big(-\frac{x(S)}{2}\Big)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\Big].\end{aligned}$$

The last equation arises because $\lambda > 0$.

- If $\mu > 0$ and $\beta > 0$, equations (5.4), (5.5) and (5.14) imply that

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \lim_{U\to\infty} \Bigg\{ \frac{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} M_{k,m}(x(U))}{U^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x(U)} W_{k,m}(x(U))} \frac{1}{\delta\sqrt{|\beta\mu|}} \\
& \Big[U^{1/2} \exp\Big(-\frac{x(U)}{2}\Big) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
& + K U^{-1/2} \exp\Big(-\frac{x(U)}{2}\Big) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \\
& - S^{1/2} \exp\Big(-\frac{x(S)}{2}\Big) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& - K S^{-1/2} \exp\Big(-\frac{x(S)}{2}\Big) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Big] \Bigg\} \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{M_{k,m}(x(U))}{W_{k,m}(x(U))} \exp\Big(-\frac{x(U)}{2}\Big) U^{1/2} \Big[W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
& + K U^{-1} W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Big[S^{1/2} \exp\Big(-\frac{x(S)}{2}\Big) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + K S^{-1/2} \exp\Big(-\frac{x(S)}{2}\Big) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Big] \lim_{U\to\infty} \frac{M_{k,m}(x(U))}{W_{k,m}(x(U))} \; .
\end{aligned}
$$

If $\beta > 0$ then $\lim_{U\to\infty} x(U) = 0$. Then, if we apply this result and equations (13.1.32) and (13.1.33) of Abramowitz and Stegun (1972, p.505), the last equation becomes

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{\frac{1}{2}+m}\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{\frac{1}{2}+m}\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)} \right. \\
& \exp\bigg(-\frac{x(U)}{2}\bigg)U^{1/2}\bigg[\exp\bigg(-\frac{x(U)}{2}\bigg)\big(x(U)\big)^m\mathcal{U}\bigg(\frac{1}{2}+m-k,2m,x(U)\bigg) \\
& \left. +KU^{-1}\exp\bigg(-\frac{x(U)}{2}\bigg)\big(x(U)\big)^{1+m}\mathcal{U}\bigg(\frac{3}{2}+m-k,2m+2,x(U)\bigg)\bigg]\right\} \\
& -\frac{1}{\delta\sqrt{|\beta\mu|}}\bigg[S^{1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& +KS^{-1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\bigg] \\
& \lim_{U\to\infty} \frac{e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{\frac{1}{2}+m}\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{e^{-\frac{1}{2}x(U)}\big(x(U)\big)^{\frac{1}{2}+m}\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)} ,
\end{aligned}
$$

where $\mathcal{M}(a,b,x)$ and $\mathcal{U}(a,b,x)$ are the Kummer's functions.

Simplifying the right-hand side of the last equation, it follows that

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} e^{x(U)} \lim_{U\to\infty} \left\{ \frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)} U^{1/2} \right. \\
& \bigg[\big(x(U)\big)^m\mathcal{U}\bigg(\frac{1}{2}+m-k,2m,x(U)\bigg) \\
& \left. +KU^{-1}\big(x(U)\big)^{1+m}\mathcal{U}\bigg(\frac{3}{2}+m-k,2m+2,x(U)\bigg)\bigg]\right\} \\
& -\frac{1}{\delta\sqrt{|\beta\mu|}}\bigg[S^{1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& +KS^{-1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\bigg] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)} .
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
= \quad &\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}U^{1/2}\right.\\
&\left[\big(x(U)\big)^m\mathcal{U}\bigg(\frac{1}{2}+m-k,2m,x(U)\bigg)\right.\\
&\left.\left.+KU^{-1}\big(x(U)\big)^{1+m}\mathcal{U}\bigg(\frac{3}{2}+m-k,2m+2,x(U)\bigg)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[S^{1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.+KS^{-1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}\ .
\end{aligned}
\tag{E.8}
$$

Using equations (13.5.6)-(13.5.10) of Abramowitz and Stegun (1972, p.508) and using the definition of m provided in equation (5.9), we can rewrite $\mathcal{U}\Big(\frac{1}{2}+m-k,1+2m,x(U)\Big)$, $\mathcal{U}\Big(\frac{1}{2}+m-k,2m,x(U)\Big)$ and $\mathcal{U}\Big(\frac{3}{2}+m-k,2m+2,x(U)\Big)$ as:

$$
\begin{aligned}
&\mathcal{U}\bigg(\frac{1}{2}+m-k,1+2m,x(U)\bigg)\\
&=\begin{cases}
\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}+O(|x(U)|^{2m-1}) & \Leftarrow \beta<\frac{1}{2}\\
\frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)}+O(|\ln x(U)|) & \Leftarrow \beta=\frac{1}{2}\\
\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}+O(1) & \Leftarrow \beta>\frac{1}{2}
\end{cases},
\end{aligned}
\tag{E.9}
$$

$$
\begin{aligned}
&\mathcal{U}\bigg(\frac{1}{2}+m-k,2m,x(U)\bigg)\\
&=\begin{cases}
\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{1-2m}+O(|x(U)|^{2m-2}) & \Leftarrow \beta<\frac{1}{4}\\
\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)}\frac{1}{x(U)}+O(|\ln x(U)|) & \Leftarrow \beta=\frac{1}{4}\\
\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{1-2m}+O(1) & \Leftarrow \frac{1}{4}<\beta<\frac{1}{2}\\
-\frac{1}{\Gamma(1-k)}[\ln(x(U))+\Psi(1-k)+2\gamma]+O(x(U)\ln x(U)) & \Leftarrow \beta=\frac{1}{2}\\
\frac{\Gamma(1-2m)}{\Gamma(\frac{3}{2}-m-k)}+O(|x(U)|^{1-2m}) & \Leftarrow \beta>\frac{1}{2}
\end{cases},
\end{aligned}
\tag{E.10}
$$

and

$$\mathcal{U}\left(\frac{3}{2}+m-k, 2m+2, x(U)\right) = \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\big(x(U)\big)^{-1-2m} + O(|x(U)|^{2m}), \quad \text{(E.11)}$$

where $\Psi(x)$ is the logarithmic derivative of the gamma function and γ is the Euler's constant.

Replacing equations (E.9), (E.10) and (E.11) into equation (E.8), we get

- ► $\beta < \frac{1}{4}$

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}} U^{1/2} \\
& \Bigg[\big(x(U)\big)^m \frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{1-2m} \\
& + KU^{-1}\big(x(U)\big)^{1+m} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\big(x(U)\big)^{-1-2m}\Bigg]\Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[S^{1/2}\exp\left(-\frac{x(S)}{2}\right) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}} \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \Bigg\{\mathcal{M}\left(\frac{1}{2}+m-k, 1+2m, x(U)\right)\big(x(U)\big)^{2m} \\
& U^{1/2}\Bigg[\big(x(U)\big)^{1-m}\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} + KU^{-1}\big(x(U)\big)^{-m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\Bigg]\Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[S^{1/2}\exp\left(-\frac{x(S)}{2}\right) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}}\big(x(U)\big)^{2m}.
\end{aligned}$$

If $\beta > 0$ then $\lim_{U\to\infty} x(U) = 0$. Then, if we use this result and definition of m in equation (5.9) and equation (13.5.5) of Abramowitz and Stegun (1972, p.508), the last equation can be written as

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \Bigg\{ \big(x(U)\big)^{\frac{1}{4|\beta|}} U^{1/2} \\
& \bigg[x(U) \frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} + KU^{-1} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \bigg] \Bigg\} .
\end{aligned}
$$

Using the definition of $x(U)$ provided in equation (5.6), the last equation becomes

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) = \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \Bigg\{ \bigg(\frac{|\mu|}{\delta^2|\beta|} U^{-2\beta} \bigg)^{\frac{1}{4|\beta|}} U^{1/2} \\
& \bigg[x(U) \frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} + KU^{-1} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \bigg] \Bigg\} \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \Bigg\{ \bigg(\frac{|\mu|}{\delta^2|\beta|} \bigg)^{m} \\
& \bigg[x(U) \frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} + KU^{-1} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \bigg] \Bigg\} \\
& = 0 \, .
\end{aligned} \tag{E.12}
$$

► $\beta = \frac{1}{4}$

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
& = \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{\mathcal{M}\big(\frac{1}{2}+1-k, 3, x(U)\big)}{\frac{\Gamma(2)}{\Gamma(\frac{1}{2}+1-k)} \big(x(U)\big)^{-2}} U^{1/2} \\
& \bigg[x(U) \bigg(\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)} \frac{1}{x(U)} + |\ln x(U)| \bigg) \\
& + KU^{-1} \big(x(U)\big)^2 \frac{\Gamma(3)}{\Gamma(\frac{3}{2}+1-k)} \big(x(U)\big)^{-3} \bigg] \Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \bigg[S^{1/2} \exp\bigg(-\frac{x(S)}{2} \bigg) W_{k-\frac{1}{2},1-\frac{1}{2}}\big(x(S)\big) \\
& + KS^{-1/2} \exp\bigg(-\frac{x(S)}{2} \bigg) W_{k-\frac{1}{2},1+\frac{1}{2}}\big(x(S)\big) \bigg] \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+1-k, 3, x(U)\big)}{\frac{\Gamma(2)}{\Gamma(\frac{1}{2}+1-k)} \big(x(U)\big)^{-2}} ,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+1-k)}{\Gamma(2)} \lim_{U\to\infty} \left\{ \mathcal{M}\left(\frac{1}{2}+1-k,3,x(U)\right)\big(x(U)\big)^2 U^{1/2} \right. \\
& \left. \left[\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)} + x(U)|\ln x(U)| + KU^{-1}\big(x(U)\big)^{-1}\frac{\Gamma(3)}{\Gamma(\frac{3}{2}+1-k)}\right]\right\} \\
& -\frac{1}{\delta\sqrt{|\beta\mu|}}\left[S^{1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},1-\frac{1}{2}}\big(x(S)\big)\right. \\
& \left. +KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},1+\frac{1}{2}}\big(x(S)\big)\right] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+1-k,3,x(U)\big)}{\frac{\Gamma(2)}{\Gamma(\frac{1}{2}+1-k)}}\big(x(U)\big)^2 .
\end{aligned}
$$

If $\beta > 0$ then $\lim_{U\to\infty} x(U) = 0$. Then, if we use this result and equation (13.5.5) of Abramowitz and Stegun (1972, p.508), we get

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+1-k)}{\Gamma(2)} \lim_{U\to\infty} \left\{ \big(x(U)\big)^2 U^{1/2} \right. \\
& \left. \left[\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)} + \frac{|\ln x(U)|}{\frac{1}{x(U)}} + KU^{-1}\big(x(U)\big)^{-1}\frac{\Gamma(3)}{\Gamma(\frac{3}{2}+1-k)}\right]\right\} .
\end{aligned}
$$

Using the definition of $x(U)$ provided in equation (5.6) as well as equation (13.5.5) of Abramowitz and Stegun (1972, p.508), the last equation becomes

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+1-k)}{\Gamma(2)} \lim_{U\to\infty} \left\{ \left(\frac{|\mu|}{\delta^2|\beta|}U^{-\frac{1}{2}}\right)^2 U^{1/2} \right. \\
& \left. \left[\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)} + \frac{|\ln x(U)|}{\frac{1}{x(U)}} + KU^{-1}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-\frac{1}{2}}\right)^{-1}\frac{\Gamma(3)}{\Gamma(\frac{3}{2}+1-k)}\right]\right\} \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+1-k)}{\Gamma(2)} \lim_{U\to\infty} \left\{ \left(\frac{|\mu|}{\delta^2|\beta|}\right)^2 U^{-1/2} \right. \\
& \left. \left[\frac{\Gamma(1)}{\Gamma(\frac{3}{2}-k)} + \frac{|\ln x(U)|}{\frac{1}{x(U)}} + KU^{-1/2}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-1}\frac{\Gamma(3)}{\Gamma(\frac{3}{2}+1-k)}\right]\right\} . \qquad \text{(E.13)}
\end{aligned}
$$

Equation (E.13) is equal to zero because $\lim_{x\to 0} \frac{|\ln(x)|}{\frac{1}{x}} = 0$ (this is easy to verify using the L'Hospital's rule).

► $\frac{1}{4} < \beta < \frac{1}{2}$

$$\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \ & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}} U^{1/2} \right. \\
& \left[\big(x(U)\big)^m \left(\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} \big(x(U)\big)^{1-2m} + 1 \right) \right. \\
& \left.\left. + KU^{-1} \big(x(U)\big)^{1+m} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \big(x(U)\big)^{-1-2m} \right] \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[S^{1/2} \exp\left(-\frac{x(S)}{2} \right) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. + KS^{-1/2} \exp\left(-\frac{x(S)}{2} \right) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \right] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}}
\end{aligned}$$

Simplifying the right-hand side of the last equation, it follows that

$$\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \ & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}} (x(U))^{2m} U^{1/2} \right. \\
& \left. \left[\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)} \big(x(U)\big)^{1-m} + \big(x(U)\big)^m + KU^{-1}\big(x(U)\big)^{-m} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \right] \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[S^{1/2} \exp\left(-\frac{x(S)}{2} \right) W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. + KS^{-1/2} \exp\left(-\frac{x(S)}{2} \right) W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \right] \\
& \lim_{U\to\infty} \frac{\mathcal{M}\big(\frac{1}{2}+m-k, 1+2m, x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}} (x(U))^{2m} .
\end{aligned}$$

Using equation (13.5.5) of Abramowitz and Stegun (1972, p.508) and using the definitions of $x(U)$ and m provided in equations (5.6) and (5.9), respectively, the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)}\lim_{U\to\infty}\Bigg\{\bigg(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\bigg)^{2m}U^{1/2}\\
&\bigg[\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{1-m}+\big(x(U)\big)^m\\
&+KU^{-1}\bigg(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\bigg)^{-m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\bigg]\Bigg\}\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)}\lim_{U\to\infty}\Bigg\{\bigg(\frac{|\mu|}{\delta^2|\beta|}\bigg)^{2m}U^{-1/2}\\
&\bigg[\frac{\Gamma(2m-1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{1-m}+\big(x(U)\big)^m\\
&+KU^{-1/2}\bigg(\frac{|\mu|}{\delta^2|\beta|}\bigg)^{-m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\bigg]\Bigg\}\\
=\;& 0\,.
\end{aligned}
\tag{E.14}
$$

► $\beta=\frac{1}{2}$

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{\mathcal{M}\big(1-k,2,x(U)\big)}{\frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)}+|\ln x(U)|}U^{1/2}\\
&\bigg[\big(x(U)\big)^{\frac{1}{2}}\frac{-1}{\Gamma(1-k)}\bigg(\ln\big(x(U)\big)+\Psi(1-k)+2\gamma\bigg)\\
&+KU^{-1}\big(x(U)\big)^{1+\frac{1}{2}}\frac{\Gamma(2)}{\Gamma(2-k)}\big(x(U)\big)^{-2}\bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\bigg[S^{1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},0}\big(x(S)\big)\\
&+KS^{-1/2}\exp\bigg(-\frac{x(S)}{2}\bigg)W_{k-\frac{1}{2},1}\big(x(S)\big)\bigg]\lim_{U\to\infty}\frac{\mathcal{M}\big(1-k,2,x(U)\big)}{\frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)}+|\ln x(U)|}\,.
\end{aligned}
$$

Simplifying the right-hand side of last equation, we get

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{\mathcal{M}\big(1-k,2,x(U)\big)}{\frac{\Gamma(1)}{\Gamma(1-k)}+x(U)|\ln x(U)|}U^{1/2}x(U)\\
&\Bigg[\frac{-1}{\Gamma(1-k)}\Big(\big(x(U)\big)^{\frac{1}{2}}\ln\big(x(U)\big)+\big(x(U)\big)^{\frac{1}{2}}\Psi(1-k)+2\gamma\big(x(U)\big)^{\frac{1}{2}}\Big)\\
&+KU^{-1}\big(x(U)\big)^{-\frac{1}{2}}\frac{\Gamma(2)}{\Gamma(2-k)}\Bigg]\Bigg\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[S^{1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},0}\big(x(S)\big)\\
&+KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},1}\big(x(S)\big)\Bigg]\lim_{U\to\infty}\frac{\mathcal{M}\big(1-k,2,x(U)\big)}{\frac{\Gamma(1)}{\Gamma(1-k)}+\frac{|\ln x(U)|}{\frac{1}{x(U)}}}x(U)\,.
\end{aligned}
$$

Using the equation (13.5.5) of Abramowitz and Stegun (1972, p.508) and the definitions of $x(U)$ and m, the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)}+x(U)|\ln x(U)|}U^{1/2}\frac{|\mu|}{\delta^2|\beta|}U^{-1}\\
&\Bigg[\frac{-1}{\Gamma(1-k)}\left(\frac{\ln\big(x(U)\big)}{\big(x(U)\big)^{-\frac{1}{2}}}+\big(x(U)\big)^{\frac{1}{2}}\Psi(1-k)+2\gamma\big(x(U)\big)^{\frac{1}{2}}\right)\\
&+KU^{-1}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-1}\right)^{-\frac{1}{2}}\frac{\Gamma(2)}{\Gamma(2-k)}\Bigg]\Bigg\}\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)}+x(U)|\ln x(U)|}U^{-1/2}\frac{|\mu|}{\delta^2|\beta|}\\
&\Bigg[\frac{1}{\Gamma(1-k)}\left(\frac{-\ln\big(x(U)\big)}{\big(x(U)\big)^{-\frac{1}{2}}}-\big(x(U)\big)^{\frac{1}{2}}\Psi(1-k)-2\gamma\big(x(U)\big)^{\frac{1}{2}}\right)\\
&+KU^{-1/2}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-\frac{1}{2}}\frac{\Gamma(2)}{\Gamma(2-k)}\Bigg]\Bigg\}\\
=\;& 0\,.
\end{aligned}
\tag{E.15}
$$

The last equation arises because $\lim_{x\to 0}\frac{-\ln(x)}{(\sqrt{x})^{-1}}=0$ (this is easy to verify if we apply the L'Hospital's rule) and due to the fact that $\lim_{U\to 0}x(U)=0$.

▶ $\beta > \frac{1}{2}$

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)$$
$$=\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}U^{1/2}\right.$$
$$\left.\left[\big(x(U)\big)^m\frac{\Gamma(1-2m)}{\Gamma(\frac{3}{2}-m-k)}+KU^{-1}\big(x(U)\big)^{1+m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\big(x(U)\big)^{-1-2m}\right]\right\}$$
$$-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[S^{1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.$$
$$\left.+KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]$$
$$\lim_{U\to\infty}\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}$$
$$=\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}+(x(U))^{2m}}\big(x(U)\big)^{2m}U^{1/2}\right.$$
$$\left.\left[\big(x(U)\big)^m\frac{\Gamma(1-2m)}{\Gamma(\frac{3}{2}-m-k)}+KU^{-1}\big(x(U)\big)^{-m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\right]\right\}$$
$$-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[S^{1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.$$
$$\left.+KS^{-1/2}\exp\left(-\frac{x(S)}{2}\right)W_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]$$
$$\lim_{U\to\infty}\frac{\mathcal{M}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}+(x(U))^{2m}}\big(x(U)\big)^{2m}.$$

Using equation (13.5.5) of Abramowitz and Stegun (1972, p.508) and the definitions of $x(U)$ and m in equation (5.9), it follows that

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)$$
$$=\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}+(x(U))^{2m}}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{2m}U^{1/2}\right.$$
$$\left.\left[\big(x(U)\big)^m\frac{\Gamma(1-2m)}{\Gamma(\frac{3}{2}-m-k)}+KU^{-1}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{-m}\frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)}\right]\right\},$$

i.e.,

$$
\begin{aligned}
& \lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} + (x(U))^{2m}} \left(\frac{|\mu|}{\delta^2|\beta|}\right)^{2m} U^{-1/2} \\
& \left[\big(x(U)\big)^m \frac{\Gamma(1-2m)}{\Gamma(\frac{3}{2}-m-k)} + KU^{-1/2} \left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-m} \frac{\Gamma(2m+1)}{\Gamma(\frac{3}{2}+m-k)} \right] \Bigg\} \\
= \; & 0 \, . \qquad \text{(E.16)}
\end{aligned}
$$

From equations (E.12) - (E.16) we conclude that

$$
\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) = 0 \,, \qquad \text{(E.17)}
$$

when $\mu > 0$ and $\beta > 0$.

For the integral $J_\lambda(K,S,U)$, equation (5.15) yields:

$$
\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) \; = \; & \lim_{U\to\infty} \Bigg\{ \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[\frac{2mU^{1/2}}{k+m-\frac{1}{2}} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
& + \frac{KU^{-1/2}}{2m+1} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \\
& - \frac{2mS^{1/2}}{k+m-\frac{1}{2}} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& - \frac{KS^{-1/2}}{2m+1} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Bigg] \Bigg\} \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{2mU^{1/2}}{k+m-\frac{1}{2}} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
& + \frac{KU^{-1/2}}{2m+1} \exp\left(-\frac{x(U)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[\frac{2mS^{1/2}}{k+m-\frac{1}{2}} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + \frac{KS^{-1/2}}{2m+1} \exp\left(-\frac{x(S)}{2}\right) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \Bigg] \, .
\end{aligned}
$$

Using equation (13.1.32) of Abramowitz and Stegun (1972, p.505), the last equation

becomes

$$
\begin{aligned}
& \lim_{U\to\infty} J_\lambda(K,S,U) \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{2mU^{1/2}}{k+m-\frac{1}{2}} e^{-x(U)} \big(x(U)\big)^m \mathcal{M}\Big(\frac{1}{2}+m-k, 2m, x(U)\Big) \\
& + \frac{KU^{-1/2}}{2m+1} e^{-x(U)} \big(x(U)\big)^{1+m} \mathcal{M}\Big(\frac{3}{2}+m-k, 2+2m, x(U)\Big)\Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[\frac{2mS^{1/2}}{k+m-\frac{1}{2}} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + \frac{KS^{-1/2}}{2m+1} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg] \\
= \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} e^{-x(U)} \lim_{U\to\infty} \Bigg\{ \frac{2mU^{1/2}}{k+m-\frac{1}{2}} \Big(\frac{|\mu|}{\delta^2|\beta|} U^{-2\beta}\Big)^m \\
& \mathcal{M}\Big(\frac{1}{2}+m-k, 2m, x(U)\Big)\Bigg\} \\
& + \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{KU^{-1/2}}{2m+1} e^{-x(U)} \big(x(U)\big)^{1+m} \mathcal{M}\Big(\frac{3}{2}+m-k, 2+2m, x(U)\Big)\Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[\frac{2mS^{1/2}}{k+m-\frac{1}{2}} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + \frac{KS^{-1/2}}{2m+1} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg],
\end{aligned}
$$

where $\mathcal{M}(a,b,x)$ is the Kummer's function. The last equation arises from the definition of $x(U)$ in equation (5.6).

If $\beta > 0$ then $\lim_{U\to\infty} x(U) = 0$. Using equation (13.5.5) of (Abramowitz and Stegun, 1972, p.508), it follows that

$$
\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) = \; & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{2mU^{\frac{1}{2}-2\beta m}}{k+m-\frac{1}{2}} \Big(\frac{|\mu|}{\delta^2|\beta|}\Big)^m \Bigg\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \Bigg[\frac{2mS^{1/2}}{k+m-\frac{1}{2}} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
& + \frac{KS^{-1/2}}{2m+1} \exp\Big(-\frac{x(S)}{2}\Big) M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg].
\end{aligned}
$$

Using the definition of m provided in equation (5.9), the last equation can be written as

$$
\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) &= \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{2m}{k+m-\frac{1}{2}}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^m \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{2mS^{1/2}}{k+m-\frac{1}{2}}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
&\quad +\frac{KS^{-1/2}}{2m+1}\exp\left(-\frac{x(S)}{2}\right)M_{k-\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg].
\end{aligned}
$$

- If $\mu < 0$ and $\beta > 0$, equations (5.4), (5.5) and (5.14) imply that

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U) \\
=\;& \lim_{U\to\infty}\Bigg\{\frac{U^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(U)}M_{k,m}(x(U))}{U^{\beta+\frac{1}{2}}e^{\frac{\epsilon}{2}x(U)}W_{k,m}(x(U))}\frac{1}{\delta\sqrt{|\beta\mu|}} \\
&\Bigg[\frac{U^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
&-\frac{KU^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big) \\
&-\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
&+\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg]\Bigg\} \\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{M_{k,m}(x(U))}{W_{k,m}(x(U))}\Bigg[\frac{U^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big) \\
&-\frac{KU^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\Bigg]\Bigg\} \\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
&-\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg]\lim_{U\to\infty}\frac{M_{k,m}(x(U))}{W_{k,m}(x(U))}\,.
\end{aligned}
$$

Using equations (13.1.32) and (13.1.33) of Abramowitz and Stegun (1972, p.505), the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}\right.\\
&\left[\frac{U^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)\exp\left(-\frac{x(U)}{2}\right)\big(x(U)\big)^m\mathcal{U}\left(m-k-\frac{1}{2},2m,x(U)\right)\right.\\
&-\frac{KU^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)\exp\left(-\frac{x(U)}{2}\right)\big(x(U)\big)^{m+1}\\
&\left.\left.\mathcal{U}\left(\frac{1}{2}+m-k,2+2m,x(U)\right)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}{e^{-\frac{1}{2}x(U)}(x(U))^{\frac{1}{2}+m}\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}\\
=\;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}{\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))}U^{1/2}\big(x(U)\big)^m\right.\\
&\left[\frac{1}{k-m+\frac{1}{2}}\mathcal{U}\left(m-k-\frac{1}{2},2m,x(U)\right)\right.\\
&\left.\left.-\frac{KU^{-1}}{m+k+\frac{1}{2}}x(U)\mathcal{U}\left(\frac{1}{2}+m-k,2+2m,x(U)\right)\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{\mathcal{M}(\frac{1}{2}+m-k,1+2m,x(U))}{\mathcal{U}(\frac{1}{2}+m-k,1+2m,x(U))},
\end{aligned}
\tag{E.18}
$$

where $\mathcal{M}(a,b,x)$ and $\mathcal{U}(a,b,x)$ are the Kummer's functions.

Using equations (13.5.6) - (13.5.10) of Abramowitz and Stegun (1972, p.508) and the definition of m in equation (5.9) and given that $\lim_{U\to\infty}x(U)=0$, we can rewrite $\mathcal{U}\big(m-k-\frac{1}{2},2m,x(U)\big)$, $\mathcal{U}\big(\frac{1}{2}+m-k,2+2m,x(U)\big)$ and $\mathcal{U}\big(\frac{1}{2}+m-k,1+2m,x(U)\big)$

as:

$$\mathcal{U}\left(\frac{1}{2}+m-k,2m+2,x(U)\right)=\frac{\Gamma(2m+1)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-1-2m}+O(|x(U)|^{2m}), \quad \text{(E.19)}$$

$$\mathcal{U}\left(\frac{1}{2}+m-k,1+2m,x(U)\right) = \begin{cases} \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}+O(|x(U)|^{2m-1}) & \Leftarrow \beta<\frac{1}{2} \\ \frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)}+O(|\ln x(U)|) & \Leftarrow \beta=\frac{1}{2} \\ \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}+O(1) & \Leftarrow \beta>\frac{1}{2} \end{cases}, \quad \text{(E.20)}$$

$$\mathcal{U}\left(m-k-\frac{1}{2},2m,x(U)\right) = \begin{cases} \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}\big(x(U)\big)^{1-2m}+O(|x(U)|^{2m-2}) & \Leftarrow \beta<\frac{1}{4} \\ \frac{\Gamma(1)}{\Gamma(\frac{1}{2}-k)}\frac{1}{x(U)}+O(|\ln x(U)|) & \Leftarrow \beta=\frac{1}{4} \\ \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}\big(x(U)\big)^{1-2m}+O(1) & \Leftarrow \frac{1}{4}<\beta<\frac{1}{2} \\ -\frac{1}{\Gamma(-k)}[\ln(x(U))+\Psi(-k)+2\gamma]+O(x(U)\ln x(U)) & \Leftarrow \beta=\frac{1}{2} \\ \frac{\Gamma(1-2m)}{\Gamma(\frac{1}{2}-m-k)}+O(|x(U)|^{1-2m}) & \Leftarrow \beta>\frac{1}{2} \end{cases}, \quad \text{(E.21)}$$

where $\Psi(x)$ is the logarithmic derivative of the gamma function and γ is the Euler's constant.

Replacing equations (E.19) - (E.21) into equation (E.18) and using equation (13.5.5) of Abramowitz and Stegun (1972, p.508), we get (notice that $\lim_{U\to\infty} x(U)=0$ because $\beta>0$):

► $\beta < \frac{1}{4}$

$$
\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}} U^{1/2}\big(x(U)\big)^m \right. \\
&\left[\frac{1}{k-m+\frac{1}{2}} \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}\big(x(U)\big)^{1-2m} \right. \\
&\left.\left. -\frac{KU^{-1}}{m+k+\frac{1}{2}} x(U) \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-1-2m} \right] \right\} \\
&-\frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{S^{1/2}}{k-m+\frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \right. \\
&\left. -\frac{KS^{-1/2}}{m+k+\frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big) \right] \lim_{U\to\infty} \frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}} \,.
\end{aligned}
$$

Simplifying the right-hand side of the last equation, it follows that

$$
\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \left\{ U^{1/2}\big(x(U)\big)^m \right. \\
&\left. \left[\frac{1}{k-m+\frac{1}{2}} \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})} x(U) - \frac{KU^{-1}}{m+k+\frac{1}{2}} \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right] \right\} .
\end{aligned}
$$

Using the definitions of $x(U)$ and m provided in equations (5.6) and (5.9), respectively, the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \left\{ U^{1/2}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^m \right. \\
&\left. \left[\frac{1}{k-m+\frac{1}{2}} \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})} x(U) - \frac{KU^{-1}}{m+k+\frac{1}{2}} \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right] \right\} \\
= \quad & \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(2m)} \lim_{U\to\infty} \left\{ \left(\frac{|\mu|}{\delta^2|\beta|}\right)^m \right. \\
&\left. \left[\frac{1}{k-m+\frac{1}{2}} \frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})} x(U) - \frac{KU^{-1}}{m+k+\frac{1}{2}} \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \right] \right\} \\
= \quad & 0 \,. \qquad\qquad (E.22)
\end{aligned}
$$

► $\beta = \frac{1}{4}$

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{1}{\frac{\Gamma(2)}{\Gamma(\frac{3}{2}-k)}(x(U))^{-2}} U^{1/2} x(U) \right. \\
&\left[\frac{1}{k-1+\frac{1}{2}} \left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2}-k)} \frac{1}{x(U)} + |\ln x(U)| \right) \right. \\
&\left.\left. - \frac{KU^{-1}}{1+k+\frac{1}{2}} x(U) \frac{\Gamma(3)}{\Gamma(\frac{3}{2}-k)} \big(x(U)\big)^{-3} \right] \right\} \\
&- \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{S^{1/2}}{k-1+\frac{1}{2}} \exp\left(\frac{x(S)}{2} \right) W_{k+\frac{1}{2},1-\frac{1}{2}}\big(x(S)\big) \right. \\
&\left. - \frac{KS^{-1/2}}{1+k+\frac{1}{2}} \exp\left(\frac{x(S)}{2} \right) W_{k+\frac{1}{2},1+\frac{1}{2}}\big(x(S)\big) \right] \lim_{U\to\infty} \frac{1}{\frac{\Gamma(2)}{\Gamma(\frac{3}{2}-k)}(x(U))^{-2}} \,.
\end{aligned}$$

Using the definition of $x(U)$ provided in equation (5.6), the last equation can be written as

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{1}{\frac{\Gamma(2)}{\Gamma(\frac{3}{2}-k)}} U^{1/2} x(U) \right. \\
&\left. \left[\frac{1}{k-1+\frac{1}{2}} \left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2}-k)} x(U) + \big(x(U)\big)^2 |\ln x(U)| \right) - \frac{KU^{-1}}{1+k+\frac{1}{2}} \frac{\Gamma(3)}{\Gamma(\frac{3}{2}-k)} \right] \right\} \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{1}{\frac{\Gamma(2)}{\Gamma(\frac{3}{2}-k)}} U^{1/2} \frac{|\mu|}{\delta^2|\beta|} U^{-1/2} \right. \\
&\left. \left[\frac{1}{k-1+\frac{1}{2}} \left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2}-k)} x(U) + \big(x(U)\big)^2 |\ln x(U)| \right) - \frac{KU^{-1}}{1+k+\frac{1}{2}} \frac{\Gamma(3)}{\Gamma(\frac{3}{2}-k)} \right] \right\} \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \frac{1}{\frac{\Gamma(2)}{\Gamma(\frac{3}{2}-k)}} \frac{|\mu|}{\delta^2|\beta|} \left[\frac{1}{k-1+\frac{1}{2}} \right. \\
&\left. \lim_{U\to\infty} \left(\frac{\Gamma(1)}{\Gamma(\frac{1}{2}-k)} x(U) + \big(x(U)\big)^2 |\ln x(U)| \right) - \lim_{U\to\infty} \left\{ \frac{KU^{-1}}{1+k+\frac{1}{2}} \frac{\Gamma(3)}{\Gamma(\frac{3}{2}-k)} \right\} \right] \\
= \;& 0\,.
\end{aligned} \tag{E.23}$$

The last equation arises because $\lim_{U\to\infty} x(U) = 0$ and $\lim_{x\to 0} x^2 \ln x = 0$ (aplying the L'Hospital's rule).

► $\frac{1}{4} < \beta < \frac{1}{2}$

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U) = \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}}U^{1/2}\big(x(U)\big)^m \\
&\Bigg[\frac{1}{k-m+\frac{1}{2}}\bigg(\frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}\big(x(U)\big)^{1-2m}+1\bigg) \\
&-\frac{KU^{-1}}{m+k+\frac{1}{2}}x(U)\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-1-2m}\Bigg]\Bigg\} \\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big) \\
&-\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\Bigg]\lim_{U\to\infty}\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-2m}} \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}}U^{1/2}\big(x(U)\big)^m \\
&\Bigg[\frac{1}{k-m+\frac{1}{2}}\bigg(\frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}x(U)+\big(x(U)\big)^{2m}\bigg) \\
&-\frac{KU^{-1}}{m+k+\frac{1}{2}}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\Bigg]\Bigg\}.
\end{aligned}$$

Using the definitions of $x(U)$ and m provided in equations (5.6) and (5.9), respectively, the last equation becomes

$$\begin{aligned}
\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U) \;&=\; \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}}U^{1/2}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^m \\
&\Bigg[\frac{1}{k-m+\frac{1}{2}}\bigg(\frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}x(U)+\big(x(U)\big)^{2m}\bigg) \\
&-\frac{KU^{-1}}{m+k+\frac{1}{2}}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\Bigg]\Bigg\} \\
&=\; \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\Bigg\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^m \\
&\Bigg[\frac{1}{k-m+\frac{1}{2}}\bigg(\frac{\Gamma(2m-1)}{\Gamma(m-k-\frac{1}{2})}x(U)+\big(x(U)\big)^{2m}\bigg) \\
&-\frac{KU^{-1}}{m+k+\frac{1}{2}}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\Bigg]\Bigg\} \\
&=\; 0\,. \qquad\qquad (E.24)
\end{aligned}$$

Equation (E.24) arises because $\lim_{U\to\infty} x(U) = 0$.

► $\beta = \frac{1}{2}$

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
&= \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)} + |\ln x(U)|} U^{1/2}\big(x(U)\big)^{\frac{1}{2}} \\
&\quad \Bigg[\frac{1}{k}\left(-\frac{1}{\Gamma(-k)}\left[\ln\big(x(U)\big) + \Psi(-k) + 2\gamma\right]\right) \\
&\quad -\frac{KU^{-1}}{1+k} x(U) \frac{\Gamma(2)}{\Gamma(1-k)}\big(x(U)\big)^{-2}\Bigg]\Bigg\} \\
&\quad -\frac{1}{\delta\sqrt{|\beta\mu|}}\Bigg[\frac{S^{1/2}}{k}\exp\left(\frac{x(S)}{2}\right) W_{k+\frac{1}{2},0}\big(x(S)\big) \\
&\quad -\frac{KS^{-1/2}}{1+k}\exp\left(\frac{x(S)}{2}\right) W_{k+\frac{1}{2},1}\big(x(S)\big)\Bigg] \lim_{U\to\infty} \frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)}\frac{1}{x(U)} + |\ln x(U)|} \\
&= \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)} + x(U)|\ln x(U)|} U^{1/2}\big(x(U)\big)^{\frac{3}{2}} \\
&\quad \Bigg[\frac{1}{k}\left(- \frac{1}{\Gamma(-k)}\left[\ln\big(x(U)\big) + \Psi(-k) + 2\gamma\right]\right) - \frac{KU^{-1}}{1+k}\big(x(U)\big)^{-1}\frac{\Gamma(2)}{\Gamma(1-k)}\Bigg]\Bigg\}.
\end{aligned}$$

Using the definitions of $x(U)$ and m provided in equations (5.6) and (5.9), respectively, it follows that

$$\begin{aligned}
&\lim_{U\to\infty} \frac{\phi_\lambda(U)}{\psi_\lambda(U)} I_\lambda(K,S,U) \\
&= \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)} + x(U)|\ln x(U)|} U^{1/2}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-1}\right)^{\frac{3}{2}} \\
&\quad \Bigg[\frac{1}{k}\left(- \frac{1}{\Gamma(-k)}\left[\ln\big(x(U)\big) + \Psi(-k) + 2\gamma\right]\right) \\
&\quad -\frac{KU^{-1}}{1+k}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-1}\right)^{-1}\frac{\Gamma(2)}{\Gamma(1-k)}\Bigg]\Bigg\} \\
&= \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \Bigg\{ \frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)} + x(U)|\ln x(U)|} \left(\frac{|\mu|}{\delta^2|\beta|}\right)^{\frac{3}{2}} \\
&\quad \Bigg[\frac{1}{k}\left(- \frac{U^{-1}}{\Gamma(-k)}\ln\big(x(U)\big) - \frac{U^{-1}}{\Gamma(-k)}\left[\Psi(-k) + 2\gamma\right]\right) \\
&\quad -\frac{KU^{-1}}{1+k}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-1}\frac{\Gamma(2)}{\Gamma(1-k)}\Bigg]\Bigg\}.
\end{aligned}$$

Using again the definition of $x(U)$, the last equation becomes

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{1}{\frac{\Gamma(1)}{\Gamma(1-k)}+\frac{|\ln(x)|}{\frac{1}{x(U)}}}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{\frac{3}{2}}\right.\\
&\left[\frac{1}{k}\left(-\frac{\delta^2|\beta|}{\Gamma(-k)|\mu|}\frac{\ln(x)}{\frac{1}{x(U)}}-\frac{U^{-1}}{\Gamma(-k)}\left[\Psi(-k)+2\gamma\right]\right)\right.\\
&\left.\left.-\frac{KU^{-1}}{1+k}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-1}\frac{\Gamma(2)}{\Gamma(1-k)}\right]\right\}\\
=\;&0\,. \qquad (E.25)
\end{aligned}$$

The last equation arises because $\lim_{x\to 0}\frac{|\ln(x)|}{\frac{1}{x(U)}}=0$ and $\lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x(U)}}=0$ (these are easy to verify using the L'Hospital Rule).

- $\beta>\frac{1}{2}$

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}U^{1/2}\big(x(U)\big)^m\right.\\
&\left[\frac{1}{k-m+\frac{1}{2}}\frac{\Gamma(1-2m)}{\Gamma(\frac{1}{2}-m-k)}\right.\\
&\left.\left.-\frac{KU^{-1}}{m+k+\frac{1}{2}}x(U)\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\big(x(U)\big)^{-1-2m}\right]\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{S^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.-\frac{KS^{-1/2}}{m+k+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)W_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]\\
&\lim_{U\to\infty}\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}\\
=\;&\frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}U^{1/2}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^m\right.\\
&\left[\frac{1}{k-m+\frac{1}{2}}\frac{\Gamma(1-2m)}{\Gamma(\frac{1}{2}-m-k)}\right.\\
&\left.\left.-\frac{KU^{-1}}{m+k+\frac{1}{2}}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\left(\frac{|\mu|}{\delta^2|\beta|}U^{-2\beta}\right)^{-2m}\right]\right\}.
\end{aligned}$$

The last equation arises from definition of $x(U)$ provided in equation (5.6). Using the definition of m given in equation (5.9), it follows that

$$\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{1}{\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}(x(U))^{-2m}+1}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^m\right.\\
&\left.\left[\frac{1}{k-m+\frac{1}{2}}\frac{\Gamma(1-2m)}{\Gamma(\frac{1}{2}-m-k)}-\frac{K}{m+k+\frac{1}{2}}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^{-2m}\right]\right\}\\
=\;& 0\,. \qquad\qquad (E.26)
\end{aligned}$$

From equations (E.22) - (E.26) we conclude that

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)=0\,, \qquad (E.27)$$

when $\mu<0$ and $\beta>0$.

For the integral $J_\lambda(K,S,U)$, and using equation (5.15)

$$\begin{aligned}
&\lim_{U\to\infty}J_\lambda(K,S,U)\\
=\;& \lim_{U\to\infty}\left\{\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{2mU^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\right.\right.\\
&+\frac{KU^{-1/2}}{2m+1}\exp\left(\frac{x(U)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\\
&-\frac{2mS^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\\
&\left.\left.-\frac{KS^{-1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right]\right\}\\
=\;& \frac{1}{\delta\sqrt{|\beta\mu|}}\lim_{U\to\infty}\left\{\frac{2mU^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(U)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(U)\big)\right.\\
&\left.+\frac{KU^{-1/2}}{2m+1}\exp\left(\frac{x(U)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(U)\big)\right\}\\
&-\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{2mS^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right.\\
&\left.+\frac{KS^{-1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned}$$

Using equation (13.1.32) of Abramowitz and Stegun (1972, p.505), we know that $M_{k,m}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+m} \mathcal{M}(\frac{1}{2} + m - k, 1 + 2m, z)$, where $\mathcal{M}(a, b, x)$ is the Kummer's function. Then, the last equation becomes

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K, S, U) \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{2mU^{1/2}}{k - m + \frac{1}{2}} \big(x(U)\big)^m \mathcal{M}\bigg(-\frac{1}{2} + m - k, 2m, x(U)\bigg) \right. \\
& \left. + \frac{KU^{-1/2}}{2m+1} \big(x(U)\big)^{1+m} \mathcal{M}\bigg(\frac{1}{2} + m - k, 2 + 2m, x(U)\bigg) \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{2mS^{1/2}}{k - m + \frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m-\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. + \frac{KS^{-1/2}}{2m+1} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m+\frac{1}{2}}\big(x(S)\big) \right] \\
= \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{2mU^{1/2}}{k - m + \frac{1}{2}} \left(\frac{|\mu|}{\delta^2|\beta|} U^{-2\beta} \right)^m \mathcal{M}\bigg(-\frac{1}{2} + m - k, 2m, x(U)\bigg) \right\} \\
& + \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{KU^{-1/2}}{2m+1} \big(x(U)\big)^{1+m} \mathcal{M}\bigg(\frac{1}{2} + m - k, 2 + 2m, x(U)\bigg) \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{2mS^{1/2}}{k - m + \frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m-\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. + \frac{KS^{-1/2}}{2m+1} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m+\frac{1}{2}}\big(x(S)\big) \right].
\end{aligned}$$

The last equation arises from the definition of $x(U)$ in equation (5.6).

If $\beta > 0$ then $\lim_{U\to\infty} x(U) = 0$. Using equation (13.5.5) of Abramowitz and Stegun (1972, p.508), we get

$$\begin{aligned}
\lim_{U\to\infty} J_\lambda(K, S, U) = \;& \frac{1}{\delta\sqrt{|\beta\mu|}} \lim_{U\to\infty} \left\{ \frac{2mU^{\frac{1}{2}-2\beta m}}{k - m + \frac{1}{2}} \left(\frac{|\mu|}{\delta^2|\beta|} \right)^m \right\} \\
& - \frac{1}{\delta\sqrt{|\beta\mu|}} \left[\frac{2mS^{1/2}}{k - m + \frac{1}{2}} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m-\frac{1}{2}}\big(x(S)\big) \right. \\
& \left. + \frac{KS^{-1/2}}{2m+1} \exp\left(\frac{x(S)}{2}\right) M_{k+\frac{1}{2}, m+\frac{1}{2}}\big(x(S)\big) \right].
\end{aligned}$$

Using the definition of m provided in equation (5.9) the exponent $\frac{1}{2} - 2\beta m$ is equal to zero. Therefore, we get

$$
\begin{aligned}
\lim_{U\to\infty} J_\lambda(K,S,U) \;\; = \;\; & \frac{1}{\delta\sqrt{|\beta\mu|}}\frac{2m}{k-m+\frac{1}{2}}\left(\frac{|\mu|}{\delta^2|\beta|}\right)^m \\
& -\frac{1}{\delta\sqrt{|\beta\mu|}}\left[\frac{2mS^{1/2}}{k-m+\frac{1}{2}}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m-\frac{1}{2}}\big(x(S)\big)\right. \\
& \left.+\frac{KS^{-1/2}}{2m+1}\exp\left(\frac{x(S)}{2}\right)M_{k+\frac{1}{2},m+\frac{1}{2}}\big(x(S)\big)\right].
\end{aligned}
$$

- If $\mu = 0$ and $\beta < 0$, equations (5.4), (5.5) and (5.14) yield:

$$
\begin{aligned}
& \lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U) \\
= \;\; & \lim_{U\to\infty}\left(\frac{U^{1/2}\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}{U^{1/2}\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}\frac{2}{\delta\sqrt{2\lambda}}\Big[U^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)\right. \\
& -KU^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big) - S^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big) \\
& \left.+KS^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\right) \\
= \;\; & \frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left(\frac{\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}{\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}\Big[U^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)\right. \\
& \left.-KU^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big)\Big]\right) - \frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big) \\
& -KS^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\left(\frac{\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}{\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}\right).
\end{aligned}
$$

Given that $\lim_{U\to\infty} z(U) = +\infty$ (when $\beta < 0$) and using the asymptotic properties of the modified Bessel functions (see Abramowitz and Stegun, 1972, p.377-378), the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left(\frac{\sqrt{\frac{\pi}{2\sqrt{2\lambda}z(U)}}e^{-\sqrt{2\lambda}z(U)}}{\frac{e^{\sqrt{2\lambda}z(U)}}{\sqrt{2\pi\sqrt{2\lambda}z(U)}}}\left[U^{1/2-\beta}\frac{e^{\sqrt{2\lambda}z(U)}}{\sqrt{2\pi\sqrt{2\lambda}z(U)}}\right.\right.\\
&\left.\left.-KU^{-1/2-\beta}\frac{e^{\sqrt{2\lambda}z(U)}}{\sqrt{2\pi\sqrt{2\lambda}z(U)}}\right]\right)-\frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\\
&-KS^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\left(\frac{\sqrt{\frac{\pi}{2\sqrt{2\lambda}z(U)}}e^{-\sqrt{2\lambda}z(U)}}{\frac{e^{\sqrt{2\lambda}z(U)}}{\sqrt{2\pi\sqrt{2\lambda}z(U)}}}\right)\\
=\;& \frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left(\frac{\sqrt{2\pi^2\sqrt{2\lambda}z(U)}}{\sqrt{2\sqrt{2\lambda}z(U)}e^{2\sqrt{2\lambda}z(U)}}\frac{e^{\sqrt{2\lambda}z(U)}}{\sqrt{2\pi\sqrt{2\lambda}z(U)}}U^{1/2-\beta}\Big[1-KU^{-1}\Big]\right)\\
&-\frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\\
&\lim_{U\to\infty}\left(\frac{\sqrt{2\pi^2\sqrt{2\lambda}z(U)}}{\sqrt{2\sqrt{2\lambda}z(U)}e^{2\sqrt{2\lambda}z(U)}}\right)\\
=\;& \frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left(\frac{U^{1/2-\beta}}{\sqrt{z(U)}e^{\sqrt{2\lambda}z(U)}}\right)\lim_{U\to\infty}\Big(1-KU^{-1}\Big)\\
&-\frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\frac{\pi}{e^{2\sqrt{2\lambda}z(U)}}\,.
\end{aligned}
$$

Using the definition of $z(U)$ provided in equation (5.7), the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;& \frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left(\frac{U^{1/2-\beta}}{\sqrt{\frac{1}{\delta|\beta|}U^{-\beta}}\exp\Big(\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\Big)}\right)\\
=\;& \frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left(\frac{U^{\frac{1}{2}-\frac{\beta}{2}}}{\sqrt{\frac{1}{\delta|\beta|}}\exp\Big(\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\Big)}\right)\,. \qquad \text{(E.28)}
\end{aligned}
$$

To compute the limit in last equation, we have to apply the L'Hospital rule n times such that $\beta < -\frac{1}{2n-1}$ due to the fact that using n times the L'Hospital Rule, we get

$$\lim_{U\to\infty}\left(\frac{U^{\frac{1}{2}-\frac{\beta}{2}}}{\sqrt{\frac{1}{\delta|\beta|}}\exp\left(\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\right)}\right)=\lim_{U\to\infty}c\frac{U^{\frac{1}{2}+\frac{2n-1}{2}\beta}}{\exp\left(\frac{\sqrt{2\lambda}}{\delta|\beta|}U^{-\beta}\right)}\,, \tag{E.29}$$

where c is a constant (this is easy to verify using mathematical induction).

Therefore, if $\beta < -\frac{1}{2n-1}$ then the exponent $\frac{1}{2}+\frac{2n-1}{2}\beta$ is negative and the limit (E.29) is zero.

Then equation (E.28) becomes

$$\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)=0\,.$$

For the integral $J_\lambda(K,S,U)$, equation (5.15) yields:

$$\begin{aligned}
&\lim_{U\to\infty}J_\lambda(K,S,U)\\
=\;&\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\Big[-U^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)+KU^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big)\\
&+S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big]\\
=\;&\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\Big[-U^{1/2-\beta}\sqrt{\frac{\pi}{2\sqrt{2\lambda}z(U)}}e^{-\sqrt{2\lambda}z(U)}\\
&+KU^{-1/2-\beta}\sqrt{\frac{\pi}{2\sqrt{2\lambda}z(U)}}e^{-\sqrt{2\lambda}z(U)}\Big]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big)\\
=\;&\frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left[\sqrt{\frac{1}{z(U)}}\frac{U^{1/2-\beta}}{e^{\sqrt{2\lambda}z(U)}}\Big(-1+KU^{-1}\Big)\right]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\Big)\,.
\end{aligned}$$

The last equation arises if we apply equation (9.7.2) of Abramowitz and Stegun (1972, p.378).

Using the definition of $z(U)$ provided in equation (5.7), the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U) \\
= \;& \frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left[\sqrt{\frac{1}{\frac{1}{\delta|\beta|}U^{-\beta}}}\frac{U^{1/2-\beta}}{\exp(\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta})}\Big(-1+KU^{-1}\Big)\right] \\
&+\frac{2}{\delta\sqrt{2\lambda}}\left(S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\right) \\
= \;& \frac{2}{\delta\sqrt{2\lambda}}\sqrt{\frac{\pi}{2\sqrt{2\lambda}}}\lim_{U\to\infty}\left[\frac{U^{\frac{1}{2}-\frac{\beta}{2}}}{\sqrt{\frac{1}{\delta|\beta|}}\exp(\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta})}\Big(-1+KU^{-1}\Big)\right] \\
&+\frac{2}{\delta\sqrt{2\lambda}}\left(S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\right) \\
= \;& \frac{2}{\delta\sqrt{2\lambda}}\left(S^{1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\right).
\end{aligned}
\tag{E.30}
$$

The last equation arises using the same arguments used in equation (E.29).

- If $\mu = 0$ and $\beta > 0$, equations (5.4), (5.5) and (5.14) yield:

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U) \\
= \;& \lim_{U\to\infty}\left(\frac{U^{1/2}\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}{U^{1/2}\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}\frac{2}{\delta\sqrt{2\lambda}}\Big[U^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big)\right. \\
&-KU^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)-S^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big) \\
&\left.+KS^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big]\right) \\
= \;& \frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left(\frac{\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}{\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}\Big[U^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big)\right. \\
&\left.-KU^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)\Big]\right)-\frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big) \\
&-KS^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\left(\frac{\mathcal{I}_\nu(\sqrt{2\lambda}z(U))}{\mathcal{K}_\nu(\sqrt{2\lambda}z(U))}\right).
\end{aligned}
$$

Given that $\lim_{U\to\infty} z(U) = 0$ (when $\beta > 0$) and using the limiting forms for small arguments of the modified Bessel functions (see Abramowitz and Stegun, 1972, p.375), the last equation becomes

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left(\frac{\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^\nu}{\Gamma(\nu+1)}}{\frac{1}{2}\Gamma(\nu)(\frac{1}{2}\sqrt{2\lambda}z(U))^{-\nu}}\left[U^{1/2-\beta}\frac{1}{2}\Gamma(\nu-1)\Big(\frac{1}{2}\sqrt{2\lambda}z(U)\Big)^{-\nu+1}\right.\right.\\
&\left.\left.-KU^{-1/2-\beta}\frac{1}{2}\Gamma(\nu+1)\left(\frac{1}{2}\sqrt{2\lambda}z(U)\right)^{-\nu-1}\right]\right)-\frac{2}{\delta\sqrt{2\lambda}}\Big[S^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\\
&-KS^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\left(\frac{\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^\nu}{\Gamma(\nu+1)}}{\frac{1}{2}\Gamma(\nu)(\frac{1}{2}\sqrt{2\lambda}z(U))^{-\nu}}\right)\\
=\;&\frac{4}{\delta\sqrt{2\lambda}\Gamma(\nu)\Gamma(\nu+1)}\lim_{U\to\infty}\left(\left(\frac{1}{2}\sqrt{2\lambda}z(U)\right)^{2\nu}\frac{1}{2}U^{1/2-\beta}\Big(\frac{1}{2}\sqrt{2\lambda}z(U)\Big)^{-\nu-1}\right.\\
&\left.\left[\Gamma(\nu-1)\Big(\frac{1}{2}\sqrt{2\lambda}z(U)\Big)^2-KU^{-1}\Gamma(\nu+1)\right]\right)\\
&-\frac{4}{\delta\sqrt{2\lambda}\Gamma(\nu)\Gamma(\nu+1)}\Big[S^{1/2-\beta}\mathcal{K}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\\
&-KS^{-1/2-\beta}\mathcal{K}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big]\lim_{U\to\infty}\left(\frac{1}{2}\sqrt{2\lambda}z(U)\right)^{2\nu}.
\end{aligned}
$$

Using the definitions of $z(U)$ and ν provided in equations (5.7) and (5.11), respectively, it follows that

$$
\begin{aligned}
&\lim_{U\to\infty}\frac{\phi_\lambda(U)}{\psi_\lambda(U)}I_\lambda(K,S,U)\\
=\;&\frac{2}{\delta\sqrt{2\lambda}\Gamma(\nu)\Gamma(\nu+1)}\lim_{U\to\infty}\left(\left(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\right)^{\frac{1}{2|\beta|}-1}U^{1/2-\beta}\right.\\
&\left.\left[\Gamma(\nu-1)\Big(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\Big)^2-KU^{-1}\Gamma(\nu+1)\right]\right)\\
=\;&\frac{2}{\delta\sqrt{2\lambda}\Gamma(\nu)\Gamma(\nu+1)}\left(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}\right)^{\frac{1}{2|\beta|}-1}\\
&\lim_{U\to\infty}\left(\Gamma(\nu-1)\Big(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta}\Big)^2-KU^{-1}\Gamma(\nu+1)\right)\\
=\;&0\,.
\end{aligned}
$$

For the integral $J_\lambda(K,S,U)$, and using equation (5.15), we have

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\Big[-U^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(U)\big)+KU^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(U)\big)\\
&+S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big]. \qquad \text{(E.31)}
\end{aligned}$$

Using equation (9.6.7) of Abramowitz and Stegun (1972, p.375), we get

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\Big[-U^{1/2-\beta}\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^{\nu-1}}{\Gamma(\nu)}+KU^{-1/2-\beta}\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^{\nu+1}}{\Gamma(\nu+2)}\Big]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left[U^{-1/2-\beta}\left(-U\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^{\nu-1}}{\Gamma(\nu)}+K\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^{\nu+1}}{\Gamma(\nu+2)}\right)\right]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big).
\end{aligned}$$

Using the definitions of $z(U)$ and ν provided in equations (5.7) and (5.11), respectively, the last equation becomes

$$\begin{aligned}
&\lim_{U\to\infty} J_\lambda(K,S,U)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left[U^{-1/2-\beta}\left(-U\frac{(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}U^{-\beta})^{\frac{1}{2|\beta|}-1}}{\Gamma(\nu)}+K\frac{(\frac{1}{2}\sqrt{2\lambda}z(U))^{\nu+1}}{\Gamma(\nu+2)}\right)\right]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\lim_{U\to\infty}\left[\frac{\left(\frac{1}{2}\sqrt{2\lambda}\frac{1}{\delta|\beta|}\right)^{\nu-1}}{\Gamma(\nu)}+KU^{-1/2-\beta}\frac{\left(\frac{1}{2}\sqrt{2\lambda}z(U)\right)^{\nu+1}}{\Gamma(\nu+2)}\right]\\
&+\frac{2}{\delta\sqrt{2\lambda}}\Big(S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\Big)\\
=\ &\frac{2}{\delta\sqrt{2\lambda}}\left[\frac{1}{\Gamma(\nu)}\left(\frac{\sqrt{2\lambda}}{2\delta|\beta|}\right)^{\nu-1}+S^{1/2-\beta}\mathcal{I}_{\nu-1}\big(\sqrt{2\lambda}z(S)\big)\right.\\
&\left.-KS^{-1/2-\beta}\mathcal{I}_{\nu+1}\big(\sqrt{2\lambda}z(S)\big)\right]. \qquad \text{(E.32)}
\end{aligned}$$

□

Bibliography

Abate, J. and W. Whitt, 1995, Numerical Inversion of Laplace Transforms of Probability Distributions, *ORSA Journal on Computing* 7, 36--43.

Abramowitz, M. and I. Stegun, 1972, *Handbook of Mathematical Functions* (Dover Publications, New York).

Babbs, S., 2000, Binomial valuation of lookback options, *Journal of Economic Dynamics & Control* 24, 1499--1525.

Bekaert, G. and G. Wu, 2000, Asymmetric Volatility and Risk in Equity Markets, *Review of Financial Studies* 13, 1--42.

Benson, R. and N. Daniel, 1991, Up, Over and Out, *Risk* 4.

Bermin, H., 2000, Hedging lookback and partial lookback options using Malliavin calculus, *Applied Mathematical Finance* 7, 75--100.

Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637--654.

Borodin, A. and P. Salminen, 1996, *Handbook of Brownian Motion - Facts and Formulae*, Probability and Its Applications (Birkhauser Verlag, Basel).

Boyle, P. and S. Lau, 1994, Bumping up against the Barrier with the Bumping up against the Barrier with the Binomial Method, *Journal of Derivatives.*

Boyle, P. and Y. Tian, 1999, Pricing Lookback and Barrier Options under the CEV Process, *Journal of Financial and Quantitative Analysis* 34, 241--264.

Cheuk, T. and T. Vorst, 1997, Currency lookback options and observation frequency: a binomial approach, *Journal of International Money and Finance* 16, 173--187.

Conze, A. and R. Viswanathan, 1991, Path-dependent options: The case of lookback options, *Journal of Finance* 46, 1893--1907.

Cox, J., 1975, Notes on Option Pricing I: Constant Elasticity of Variance Diffusions, Working paper, Stanford University (reprinted in Journal of Portfolio Management, 1996, December, 15-17).

Cox, J. and S. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, 145--166.

Cox, J., S. Ross, and M. Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229--263.

Davydov, D. and V. Linetsky, 2001, Pricing and Hedging Path-Dependent Options Under the CEV Process, *Management Science* 47, 949--965.

Davydov, D. and V. Linetsky, 2003, Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach, *Operations Research* 51, 185--209.

Dennis, P. and S. Mayhew, 2002, Risk-Neutral Skewness: Evidence from Stock Options, *Journal of Financial and Quantitative Analysis* 37, 471--493.

Duffie, D., 1996, Swap Rates and Credit Quality, *Journal of Finance* 51, 921--949.

Emanuel, D. and J. MacBeth, 1982, Further Results on the Constant Elasticity of Variance Call Option Pricing Model, *Journal of Financial and Quantitative Analysis* 17, 533--554.

Eriksson, Jonatan, 2006, Explicit Pricing Formulas for Turbo Warrants, *Risk Magazine.*

Eriksson, Jonatan and Jonas Persson, 2006, Pricing Turbo Warrants, Working paper, Swedish Graduate School in Mathematics and Computing.

Faulhaber, O., Analytic Methods for Pricing Double Barrier Options in the Presence of Stochastic Volatility, Master's thesis, University of Kaiserslautern 2002.

Ferreira, S., Barrier options under the CEV diffusion, Master's thesis, ISCTE and FCUL 2009.

Geman, H. and M. Yor, 1996, Pricing and Hedging Double Barrier Options: A Probabilistic Approach, *Mathematical Finance* 6, 365--378.

Goldman, M., H. Sosin, and L. Shepp, 1979a, On contingent claims that insure ex-post optimal stock market timing, *Journal of Finance* 34, 401--413.

Goldman, M., H. Sosin, and M. Gatto, 1979b, Path-dependent options, *Journal of Finance* 34, 1111--1127.

Griebsch, S. and U. Wystup, 2008, On the Valuation of Fader and Discrete Barrier Options in Heston's Stochastic Volatility Model, Working paper, Frankfurt School of Finance & Management.

Heston, S., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327--343.

Hudson, M., 1991, The Value in Going Out, *Risk* 4, 29 -- 33.

Hull, J., 2002, *Options, Futures and Other Derivatives*, fifth ed. (Prentice-Hall).

Jackwerth, J. and M. Rubinstein, 2001, Recovering Stochastic Processes from Option Prices, Working paper, University of Konstanz and Haas School of Business.

Kuan, G. and N. Webber, 2003, Pricing Barrier Options with One-Factor Interest Rate Models, *Journal of Derivatives* 10, 33--50.

Kunitomo, N. and M. Ikeda, 1992, Pricing Options with Curved Boundaries, *Mathematical Finance* 2, 275--298.

Linetsky, V., 2004, Lookback options and diffusion hitting times: A spectral expansion approach, *Finance and Stochastics* 8, 373--398.

Lipton, A., 2001, *Mathematical Methods for Foreign Exchange*, first ed. (World Scientific, London).

Lo, C. F. and C. H. Hui, 2007, Valuing double barrier options with time-dependent parameters by Fourier series expansion, *IAENG International Journal of Applied Mathematics* 36.

Lo, C. F., H. Tang, K. C. Ku, and C. H. Hui, 2009, Valuing Time-Dependent CEV Barrier Options, *Journal of Applied Mathematics and Decision Sciences.*

MacBeth, J. and L. Merville, 1980, Tests of the Black-Scholes and Cox Call Option Valuation Models, *Journal of Finance*, 285--301.

Merton, R., 1973, The Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141--183.

Mijatović, A., 2010, Local Time and the Pricing of Time-Dependent Barrier Options, *Finance and Stochastics* 14, 13--48.

Nunes, J. and J. Dias, 2011, The Valuation of Double Barrier Options under Multifactor Pricing Models, Working paper, BRU-UNIDE and ISCTE-IUL Business School.

Pelsser, A., 2000, Pricing Double Barrier Options Using Laplace Transforms, *Finance and Stochastics* 4, 95--104.

Rubinstein, M. and E. Reiner, 1991, Breaking Down the Barriers, *Risk* 4, 28--35.

Schmalensee, R. and R. Trippi, 1978, Common Stock Volatility Expectations Implied by Option Premia, *Journal of Finance* 33, 129--147.

Schroder, M., 2000, On the valuation of double-barrier options: Computational aspects, *Journal of Computational Finance* 3.

Snyder, Gerard L., 1969, Alternative Forms of Options, *Financial Analysts Journal* 25, 93--99.

Wong, Hoi Ying and Chun Man Chan, 2008, Turbo Warrants under Stochastic Volatility, *Quantitative Financetive* 8, 739--751.

Wong, Hoi Ying and Ka Yung Lau, 2008, Analytical Valuation of Turbo Warrants under Double Exponential Jump Diffusion, *Journal of Derivatives* 15, 61--73.

Xu, C. and Y. Kwok, 2005, Integral Price Formulas for Lookback Options, *Journal of Applied Mathematics*, 117--125.

Zhang, P., 1998, *Exotic Options: A Guide to Second Generation Options*, second ed. (World Scientific, Singapore).